PREFACE.

THIS book contains the principles of the sciences of Motion and Force. These sciences were formerly treated of under the title *Mechanics*; but in recent works the name of *Kinematics* is given to the science of motion, and *Dynamics* to that of force.

Since the publication of Professors Thomson and Tait's *Natural Philosophy*, and of Tait's *Recent Advances in Physical Science*, it has been generally acknowledged that the true foundation of the science of Dynamics is the Laws of Motion, as stated by Newton. Availing myself of these works, and guided to some extent by the course followed in recent examination papers, I have attempted, with the aid of elementary mathematics only, to give in the following pages the statement and demonstration of the principles of Kinematics and Dynamics. A chapter is devoted to each of the Laws of Motion, which are explained at length and illustrated by examples. From these laws the propositions relating to forces in equilibrium follow at once as corollaries; and thus the laboured though ingenious proofs from first principles of the fundamental proposition of Statics—such as Duchayla's—are rendered unnecessary.

The subjects treated of in these pages afford an excellent opportunity to the student for the applica-

tion of the abstract principles of geometry and algebra, which he has learned. For this reason alone they deserve a place in the curriculum even of elementary schools. As a means of mental culture they are not inferior to the pure mathematics; and throughout the work I have aimed at making their study an intellectual training.

Numerous exercises of a varied character are appended to each chapter. They are intended to illustrate the subjects of the chapter, and they can all be solved by the principles explained in the text. Typical problems under each chapter have been worked as examples of the methods that the students may apply to the others. Hints for the solution of all the remaining exercises are appended to the work. In these hints such help is afforded as a teacher would give to his pupils; and thus where the aid of a teacher cannot be obtained, an intelligent student who has read the text may work through all the problems without the assistance of a master.

A great many of the exercises have been selected from the Examination Papers of the Universities of London, Cambridge, Edinburgh, and Dublin, from the Woolwich Entrance Examination Papers, and from the Papers set to candidates at examinations held by the Science and Art Department, and by the National Board of Education, Ireland. Many of the questions, however, are original, and have been specially prepared to illustrate the demonstrations of the text.

The subjects have not been treated in accordance

with the programme for any particular examination. As, however, the work is designed as a text-book for use in schools, and for students who are preparing for examinations, care has been taken to include all important principles and demonstrations usually given in elementary works on Mechanics.

The articles and chapters marked thus [*] are somewhat more difficult than the others. They are given in their proper places in the text, but they may be omitted on a first reading of the work.

J. J. D.

DUBLIN,
May, 1881.

PREFACE TO THE SECOND EDITION.

SOME errors in the exercises and answers of the First Edition of this work have been corrected in the present edition.

April, 1882.

CONTENTS.

Elementary Text-Books of Natural Philosophy.

CHAPTER I.

INTRODUCTORY.

THE laws of the material universe form the subject of Natural Philosophy.

There appear to be five fundamental conceptions to which all our ideas of this universe may be referred. They are those of *Time*, *Space*, *Motion*, *Force*, and *Matter*.

TIME AND SPACE.—*Algebra* has been called the science of pure Time, and *Geometry* the science of pure Space. With these sciences we shall not be directly concerned; but we shall make use of their results and their methods.

A measure of time is supplied to all mankind by the revolution of the earth on its axis; and a *second*, which is the 86,400th part of the mean solar day, is the unit of time that is ordinarily employed.

There is no such natural measure of space, and hence different nations make use of different and arbitrary units of space. In this country the scientific unit of length generally adopted is the *foot*, which is one-third of the *standard yard*. The standard yard is defined by Act of Parliament to be the distance between the centres of the transverse lines in the two gold plugs in the bronze bar deposited in the Exchequer, the temperature being 62° Fah.

The French standard of length is the *metre*. It is nearly equal to 39·37 British inches. The 100th part of the metre is the *centimetre*.

Units of surface and volume are derived from the units of length.

MOTION.—The science which treats of Motion, without considering the causes which produce it or the quantity of matter moved, has obtained the name of *Kinematics*.

FORCE.—The science which treats of Force is *Dynamics*. Dynamics is divided into *Kinetics* and *Statics*. Kinetics treats of forces producing motion, and Statics of forces maintaining rest.

MATTER.—The science of Matter is *Physics*.

Kinematics and Dynamics—the sciences of Motion and Force—form the subject of this work.

CHAPTER II.

KINEMATICS—MOTION.

Relations between Space, Time, and Velocity.

1. The motion of a body may be considered without taking account of the cause of the motion, or the matter of the body which is moved. Kinematics, the science which treats of motion in this way, is a mixed science of Time and Space, and it therefore employs the processes of Algebra and Geometry. Since in Kinematics the quantity of matter in the body is not considered, it will generally be convenient to regard the body as indefinitely small in all its dimensions. Such a body is called a *particle* or a *material point*.

2. Motion is change of position. A body is at *rest* if its situation in space remains unaltered; it is in *motion* if its position is changing.

Rest and Motion are relative terms. A book lying on the deck of a ship is at rest with respect to the ship, but in motion with respect to the earth. A stone on the ground is at rest as regards the earth, but it, along with the earth, is in motion. Absolute rest, so far as we know, does not exist in the universe; and of the absolute motion of any body we are wholly ignorant.

3. In considering the motion of a body from one position in space to another, we may also consider the *time* it takes to pass through any distance. The *rate* of the motion is called the Velocity.

Velocity may be either *uniform* or *variable*. The velocity of a moving body is uniform when it passes over equal spaces in equal times. It is variable when it passes over unequal spaces in equal times.

4. Uniform velocity is expressed by the number of units of length which the body passes over in the unit of time. Thus, if a train is moving uniformly at the rate of 5 miles in 10 minutes its velocity is 44 feet per second.

Variable velocity is expressed at any instant by the number of units of length which the body would pass over in a unit of time, if the velocity which it had at that instant continued uniform for the unit of time. If, for instance, a railway train is moving with a variable speed, and that at any instant its velocity is said to be 1 mile per minute, it is meant that if from that instant the velocity remained uniform with the velocity it had at that instant, the train would pass over the sixtieth part of a mile in one second, or a mile in one minute.

5. A variable velocity may increase or decrease and either uniformly or variably. A velocity variably altering will not be considered by us. It is convenient to have one name for a velocity either increasing or decreasing uniformly. Such a velocity is called an *accelerated* velocity; and the change of velocity in the unit of time is called the *acceleration*. The acceleration is positive when an increase, and negative when a decrease. Acceleration is then the rate of change of velocity. Thus, if a body be moving with the velocities in successive seconds of 20 feet, 25 feet, 30 feet, ... per second, it is moving with a uniformly accelerated velocity, and the acceleration is 5 feet per second *gained* per second. If, again, a body move with the velocities in successive seconds of 12 feet, 10 feet, 8 feet, ... per second, it is moving with a uniformly accelerated velocity, the acceleration being 2 feet per second *lost* per second, or an acceleration of − 2 ft. per second, per second.

6. *Relations between Space, Time, and Velocity. Uniform Velocity.*—A body moving with uniform velocity passes over the same space or length of path in each successive

second, and the space described in any second is the measure of the velocity. Thus, if a body be moving with the uniform velocity u feet per second, it will in one second describe u feet, in 2 seconds $2u$ feet, in t seconds tu feet. Calling the space described in t seconds s, then—

$$s = ut.$$

This equation expresses the relation between the space, the time, and the uniform velocity, and any two of these quantities being given the remaining one may be calculated.

7. *Uniformly Accelerated Velocity.*—Let us suppose that a body has an initial uniform velocity of u feet per second, and that an acceleration f feet per second is communicated to it in the direction of its motion.

(1) *To find its velocity at the end of t seconds.*

Since the body has at starting a velocity of u feet per second, and receives an increase of velocity of f feet per second in each second, its velocities will be—

At starting,	$= u.$
At end of 1st second,	$= u + f$
At end of 2nd second,	$= u + 2f$
At end of 3rd second,	$= u + 3f$
. . . .	. . . . . .
At end of tth second,	$= u + tf.$

Let v denote the velocity at the end of t seconds, then—

$$v = u + ft.$$

This equation expresses the relations between velocity, time, and acceleration.

8. (2) *To find the space described during the time t by a body which starts with the velocity u feet per second and receives an acceleration f feet per second.*

We shall assume that if a body move with a *uniformly accelerated* velocity the space described in any time is the same as what the body would describe if it moved during the time with a *uniform* velocity equal to the mean of its two terminal velocities. If, for example, a railway train pass a certain point on the line with a velocity of 10 miles

per hour, and moving with a uniformly accelerated velocity pass another point one hour afterwards with a velocity of 20 miles per hour, it may be assumed as almost self-evident that the two points are 15 miles apart and that the space passed over by the train during the hour while moving with its accelerated velocity is the same as what it would have described had it moved uniformly for the hour with the mean velocity $= \frac{10+20}{2} = 15$ miles per hour.

Now, by hypothesis, the body has an initial velocity u, and by equation, Art. 7, its velocity at the end of the time t is $u+ft$, therefore, its mean or average velocity is—

$$\frac{u+u+ft}{2} = u + \tfrac{1}{2}ft.$$

Now, if the body move with the uniform velocity $u+\frac{1}{2}ft$ for t seconds, the space described is, by Art. 6—

$$s = (u + \tfrac{1}{2}ft)\,t.$$
$$s = ut + \tfrac{1}{2}ft^2.$$

This equation expresses the relations between space, time, velocity, and acceleration.

9. (3) *To find the velocity acquired in moving through a given space.*

From Arts. 7 and 8,

$$v = u + ft \qquad (1)$$
$$s = ut + \tfrac{1}{2}ft^2 \qquad (2)$$

Squaring equation (1),

$$\begin{aligned} v^2 &= u^2 + 2uft + f^2t^2 \\ &= u^2 + 2f(ut + \tfrac{1}{2}ft^2) \\ &= u^2 + 2fs \text{ by equation (2)} \\ \therefore v^2 &= u^2 + 2fs \end{aligned}$$

This equation expresses the relation between the space described and the velocity acquired in passing through that space.

10. If in the foregoing cases the acceleration be in a direction opposite to the initial velocity, then calling the

direction of the latter positive, and that of the former negative, by similar demonstrations to the foregoing we should obtain the following results :—

$$v = u - ft \qquad (1)$$
$$s = ut - \tfrac{1}{2}ft^2 \qquad (2)$$
$$v^2 = u^2 - 2fs \qquad (3)$$

These, it will be seen, can be at once obtained from the former equations by writing $-f$ for f.

11. By writing these equations with the double sign $\pm$ we obtain general formulæ applicable to all cases of the motion of a body possessing an initial velocity, and an acceleration either in the same or in the opposite direction—

$$v = u \pm ft \qquad (1)$$
$$s = ut \pm \tfrac{1}{2}ft^2 \qquad (2)$$
$$v^2 = u^2 \pm 2fs \qquad (3)$$

In these equations the + sign is to be taken when the acceleration is in the same direction as the initial velocity, and the negative when it is in the opposite direction.

12. If there be no initial velocity, u in the above equations becomes 0, and the terms that contain u vanish. In this case, therefore—

$$v = ft \qquad (1)$$
$$s = \tfrac{1}{2}ft^2 \qquad (2)$$
$$v^2 = 2fs \qquad (3)$$

These equations apply to the case of an accelerated velocity where there is no initial velocity. They may be obtained directly by demonstrations similar to those of Arts. 7, 8, and 9.

[*] 13. The foregoing equations are so important that we shall give another proof of the formula $s = ut + \frac{1}{2}ft^2$, not involving the assumption of Art. 8.

A body has an initial velocity u, and an acceleration f in the direction of its motion is given to it, it is required to show that in the time t the space described is $ut + \frac{1}{2}ft^2$.

Let the time t be divided into n equal parts, each equal to $\frac{t}{n}$. Then since in a second the acceleration f is added to the velocity, an acceleration $\frac{ft}{n}$ is added in the time $\frac{t}{n}$. And since the body at starting has a velocity u, and an acceleration $\frac{ft}{n}$ is added during each of the equal intervals of time $\frac{t}{n}$, the velocities of the body at the beginning and at the end respectively of each interval will be as follows:—

Intervals—	1,	2,	3,	4,	. . .	n
Velocities at the beginning of each interval—	u,	$u+\frac{ft}{n}$,	$u+\frac{2ft}{n}$,	$u+\frac{3ft}{n}$,	.	$u+\frac{(n-1)ft}{n}$
Velocities at the end of each interval—	$u+\frac{ft}{n}$,	$u+\frac{2ft}{n}$,	$u+\frac{3ft}{n}$,	$u+\frac{4ft}{n}$,	. .	$u+\frac{nft}{n}$

Now, if the body moved uniformly during each interval with the velocity it had at the *beginning* of the interval, the spaces described during each interval would be found according to Art. 6 by multiplying the respective velocities by the interval $\frac{t}{n}$; and the whole space described by the body under this hypothesis would be found by adding these spaces together. Similarly, we could find the whole space described under the supposition that the body moved uniformly for each interval with the velocity it had at the *end* of the interval. Let s_1 denote the whole space that would be described if the body moved according to the first hypothesis, and let s_2 denote the space described under the second hypothesis. Then by Art. 6—

$$s_1 = u \times \frac{t}{n} + \left(u + \frac{ft}{n}\right)\frac{t}{n} + \left(u + \frac{2ft}{n}\right)\frac{t}{n} + \ . \ . \ + \left\{u + \frac{(n-1)ft}{n}\right\}\frac{t}{n}$$

$$s_2 = \left(u + \frac{ft}{n}\right)\frac{t}{n} + \left(u + \frac{2ft}{n}\right)\frac{t}{n} + \left(u + \frac{3ft}{n}\right)\frac{t}{n} + \ . \ + \left(u + \frac{nft}{n}\right)\frac{t}{n}$$

$$\therefore\ s_1 = \frac{ut}{n} + \left(\frac{ut}{n} + \frac{ft^2}{n^2}\right) + \left(\frac{ut}{n} + \frac{2ft^2}{n^2}\right) + \ldots + \left(\frac{ut}{n} + \frac{(n-1)ft^2}{n^2}\right)$$

$$s_2 = \left(\frac{ut}{n} + \frac{ft^2}{n^2}\right) + \left(\frac{ut}{n} + \frac{2ft^2}{n^2}\right) + \left(\frac{ut}{n} + \frac{3ft^2}{n^2}\right) + \ldots \left(\frac{ut}{n} + \frac{nft^2}{n^2}\right)$$

$$\therefore\ s_1 = \frac{nut}{n} + \frac{ft^2}{n^2}(1 + 2 + 3 + \ldots (n-1))$$

$$s_2 = \frac{nut}{n} + \frac{ft^2}{n^2}(1 + 2 + 3 + \ldots n)$$

And by finding the sum of each arithmetical series—

$$\therefore\ s_1 = ut + \frac{ft^2}{n^2} \cdot \frac{n(n-1)}{2} = ut + \frac{ft^2}{2}\left(1 - \frac{1}{n}\right)$$

$$s_2 = ut + \frac{ft^2}{n^2} \cdot \frac{n(n+1)}{2} = ut + \frac{ft^2}{2}\left(1 + \frac{1}{n}\right)$$

Now, the body, according to the original hypothesis, is moving with an accelerated velocity, and therefore during each of the equal intervals, it moves neither with the velocity it has at the beginning of the interval nor with that at the end, but with a varying velocity, which however is always intermediate between these two velocities. Hence, if s denote the space actually described by the body, s must lie between s_1 and s_2. But the smaller the intervals are made, and the larger consequently n becomes, the more nearly the values of s_1 and s_2 approach each other. And when the interval becomes indefinitely small, and n infinitely great, each of the values of s_1 and s_2 may be regarded as expressing the space described under a constantly accelerated velocity, since then the velocity remains uniform only for an infinitely short interval. But when n is infinite $\frac{1}{n} = 0$, and $s_1 = s_2 = ut + \frac{ft^2}{2}$. And the space s actually described by the body is intermediate between the values of s_1 and s_2, therefore—

$$s = ut + \tfrac{1}{2}ft^2.$$

14. *Falling Bodies.*—When a body falls freely in a *vacuum*, experiment shows that it receives a constant increase of velocity during each second of its fall. The motion of falling bodies consequently affords an illustration of the foregoing principles; and this case of uniformly accelerated velocity is so important that the acceleration is usually denoted by a special symbol g.

The value of g varies in different places. Its mean value for Great Britain is very nearly 32·2 feet per second. In the exercises which follow, it is supposed that there is no resistance from the air; and, unless when otherwise stated, the value of g is taken as 32 feet or 9·8 metres per second.

The equations for falling bodies are the same as those of Arts. 11 and 12, g being written for f.

$v = gt$	(1)	$v = u \pm gt$	(4)
$s = \frac{1}{2}gt^2$	(2)	$s = ut \pm \frac{1}{2}gt^2$	(5)
$v^2 = 2gs$	(3)	$v^2 = u^2 \pm 2gs$	(6)

15. The following theorems on bodies falling from rest may be demonstrated here. The first two propositions are true for all bodies moving from rest with a constant acceleration.

(1) *The spaces described by a body falling from rest are proportional to the squares of the numbers of seconds that the body has been falling.*

By equation (2) of preceding Art. the spaces in 1, 2, 3, 4, &c., seconds are $\frac{1}{2}g$, $\frac{1}{2}g \times 2^2$, $\frac{1}{2}g \times 3^2$, $\frac{1}{2}g \times 4^2$, . . . &c.; and these spaces are proportional to 1^2, 2^2, 3^2, 4^2, &c.

(2) *The spaces described in successive seconds by a body falling from rest are proportional to the odd numbers* 1, 3, 5, 7, *&c.*

Space in	1st sec.	$=$	$\frac{1}{2}g \times 1$
„	2nd „	$= \frac{1}{2}g \times 2^2 - \frac{1}{2}g \times 1^2 =$	$\frac{1}{2}g \times 3$
„	3rd „	$= \frac{1}{2}g \times 3^2 - \frac{1}{2}g \times 2^2 =$	$\frac{1}{2}g \times 5$

. . . .

and these spaces are proportional to the odd numbers.

(3) *The square of the number of quarter seconds during*

which a body falls from rest is approximately equal to the number of feet in the vertical distance described.

Let $x =$ N°. of quarter secs. for which the body falls; then $\frac{x}{4} =$ N°. of secs.; and by equation (2) Art. 14,

$$s = \tfrac{1}{2}g\left(\frac{x}{4}\right)^2 = 16\frac{x^2}{16} = x^2.$$

[*] Units of Velocity and Acceleration.

16. *Unit of Velocity.*—Velocity, like all other quantities, is expressed in terms of a unit of its own kind. The unit of velocity selected is that velocity with which a body describes the unit of length in the unit of time. Keeping to the units of time and space already fixed, the British unit of velocity is a velocity of one foot per second, and the metrical unit is a velocity of one centimetre per second.

Unit of Acceleration.—Similarly the unit of acceleration is that acceleration with which the unit of velocity is gained or lost in the unit of time. The British unit of acceleration is therefore a velocity of one foot per second gained or lost per second, and the metrical unit of acceleration is a velocity of one centimetre per second, per second.

17. A change in the units of time and space produces a change in the units of velocity and acceleration.

Change in Unit of Velocity.—Since the greater the space a body passes over in a given time, the greater the velocity, it follows that if while the unit of time remains the same, the unit of length be increased, the unit of velocity will be increased in the same ratio. Again, the greater the time taken to pass over a given space the less the velocity, and therefore if while the unit of length is unaltered, the unit of time be increased, the unit of velocity will be diminished in the same ratio. Therefore, when both the units of length and time vary, the unit of velocity varies directly as the unit of length and inversely as the unit of time.

Thus, when a foot is the unit of length and a second the

unit of time, the unit of velocity is 1 foot per second. Now, (1), if a yard be made the unit of length while the unit of time is unaltered; (2), if a minute be made the unit of time while the unit of length is unaltered; (3), if a yard be made the unit of length, and a minute the unit of time—then the unit of velocity is (1), three times; (2), one sixtieth; (3), one twentieth the original unit.

Change in Unit of Acceleration.—The unit of acceleration is the gain positively or negatively of the unit of velocity in the unit of time—that is, it is the unit of length in the unit of time gained in the unit of time. It follows that if the unit of time remain the same and the unit of length be changed, the unit of acceleration will be changed in the same ratio; but if the unit of length remain unaltered, and the unit of time vary, the unit of acceleration varies inversely as the *square* of the unit of time. The unit of time enters twice into the value of the unit of acceleration. If the unit of time be increased, the unit of velocity is diminished in the same ratio, and the unit of velocity is gained in a proportionally greater time, and therefore the unit of acceleration is diminished in the duplicate ratio of the unit of time.

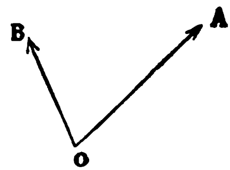
Thus, let the unit of acceleration be one foot per second, per second, the units of length and time being a foot and a second respectively. Now, (1), if a yard and a second be taken as the units; (2), if a foot and a minute be the units; (3), if a yard and a minute be the units—then the new unit of acceleration is (1), three times; (2), one thirty-six hundredth; (3), one twelve hundredth of its original value.

18. It should be remembered that the *numerical measure* of a given quantity varies inversely as the units employed. Thus, if a shilling be taken as the unit, the numerical measure of one pound sterling is 20, but if a penny be the unit, the measure is 240.

Thus, the numerical measure of a given velocity varies inversely as the unit of velocity, and the numerical measure of a given acceleration varies inversely as the unit of acceleration. Hence, the numerical measure of a velocity

varies inversely as the unit of length, and directly as the unit of time, and the numerical measure of an acceleration varies inversely as the unit of length and directly as the square of the unit of time.

19. *Representations of Velocities and Accelerations.*—Velocities and accelerations may be represented by straight lines whose lengths are proportional to the velocities or accelerations, and whose directions are those in which the body tends to move. Thus, OA and OB may represent in magnitude and direction the two velocities or the two accelerations possessed by a body at O.



EXAMPLES.

1. A body moves from rest with an acceleration of 20 yards per second: find (*a*), the space described in 10 seconds; (*b*), the velocity acquired in 10 seconds; (*c*), the velocity acquired in moving through 90 yards.

20 yards = 60 ft. = the acceleration.

We employ the equations of Art. 12, there being no initial velocity.

(*a*) $s = \frac{1}{2}ft^2 = \frac{1}{2} \times 60 \times 10^2 = 3{,}000$ feet.

(*b*) $v = ft = 60 \times 10 = 600$ ft. per 1″.

(*c*) $v^2 = 2fs \therefore v = \sqrt{2 \times 60 \times 270} = 180$ ft. per 1″.

2. A body moves from rest for 6 seconds with an acceleration of 10 feet per second: find the spaces described in the 1st second, in the 3rd second, and in the two last seconds of its motion.

We employ equations of Art. 12. The space described in the 3rd second is equal to the space described in 3 seconds, less by the space in 2 seconds, and the space in the last two seconds is equal to the whole space described, less by the space in 4 seconds.

(*a*) Space in 1st sec. $= \frac{1}{2}ft^2 = \frac{1}{2} \times 10 \times 1 = 5$ ft.

(*b*) „ 3rd „ $= \frac{1}{2}f(3^2 - 2^2) = 5(9-4) = 25$ ft.

(*c*) „ last two „ $= \frac{1}{2}f(6^2 - 4^2) = 5(36-16) = 100$ ft.

3. A body moves from rest with a uniform acceleration which ceases at the end of 5 seconds. The body continuing in motion with the velocity acquired at the end of the 5th second describes in the next 18 seconds a space of 450 metres: find the acceleration.

$\frac{450}{18}$ = vel. at end of 5th second = 25 metres per second.

$v = ft \therefore f = \frac{v}{t} = \frac{25}{5} = 5$ metres per second.

4. If the spaces described in two successive seconds by a body which has moved from rest with a uniform acceleration be 225 feet and 255 feet respectively, what will be the spaces described in the next two seconds respectively?

By Art. 15, the difference of the spaces described in two successive seconds is equal to the acceleration, $\therefore$

$$255-225=30 \text{ ft. the acceleration.} \quad 255+30=285 \text{ ft. and}$$
$$285+30=315. \quad 285 \text{ ft. and } 315 \text{ ft.}$$

5. A body falls from rest through 576 feet: find the time.

$$s=\tfrac{1}{2}gt^2 \therefore t=\sqrt{\frac{2s}{g}}=\sqrt{\frac{2\times 576}{32}}=6 \text{ secs.}$$

6. A body is thrown upward with a velocity of 96 feet per second. After how many seconds will it be moving *downward* with a velocity of 40 feet per second?

As the initial velocity is opposite in direction to the acceleration due to gravity we use equation 4 of Art. 14 with the negative sign, $v=u-gt$, where $u=96$ and $v=-40$ since it is *downward*. Therefore—

$$-40=96-32t$$
$$32t=136 \therefore t=\frac{136}{32}=4\tfrac{1}{4} \text{ secs.}$$

7. The acceleration due to gravity (32 feet per second per second) is represented by 640 when the unit of length is 5 feet: find the unit of time.

Let t seconds $=$ unit of time.

32 feet per second gained per second

$=\frac{32}{5}$ units of length per second gained per second

$=\frac{32}{5}\times t\times t$ units of length per unit of time gained per unit of time

Therefore $\frac{32t^2}{5}=640 \therefore t^2=\frac{640\times 5}{32} \therefore t=10$ seconds.

EXERCISES.

1. A body moves from rest with an acceleration of 20 ft. per second: find (*a*), the velocity in 10 seconds; (*b*), the space described in 5 seconds; (*c*), the velocity acquired in moving through 40 feet.

2. In what time will a body which moves from rest with an acceleration of 30 feet per second acquire a velocity of 120 feet per second?

3. In what time will a body moving from rest with an acceleration of 8 feet per second describe a space of 900 ft.?

4. A body possessing an initial uniform velocity of 100 feet per second has an acceleration of 10 feet per second given to it: find (*a*), the velocity in half a minute; (*b*), the space described in 5 seconds—the uniform and accelerated velocities being in the same direction.

5. If in the preceding exercise the accelerated velocity were opposite in direction to the uniform velocity, find the velocity and the space in the given times respectively.

6. A body moving with an initial velocity of 20 feet per second receives an acceleration in the opposite direction of 5 feet per second: how far will it move before it comes to rest?

7. Find in the preceding the time of the motion.

8. A body moving from rest with a uniform acceleration describes a space of 132 feet in the 6th second: find the acceleration, and the whole space described.

9. A body moves from rest with an acceleration of 80 yards per minute: what space will it describe in 30 seconds?

10. A body which has moved from rest is observed to pass over spaces of 75 feet and 105 feet in two successive seconds: what space was described in each of the two preceding seconds?

11. A body which has moved from rest is observed to pass over 27 feet and 33 feet in two successive seconds respectively: what space will it describe in the 15th second?

12. A body moving with a certain velocity loses a velocity of 15 feet per second every second, and comes to rest in half a minute: what was its velocity?

13. A body possessing a certain velocity loses an acceleration of 8 feet per second, and comes to rest after moving through 100 feet: find the velocity.

14. A stone is projected downwards from the top of a precipice with a velocity of 50 metres per second: (*a*), when will it have acquired a velocity of 99 metres per second? (*b*). What space will it have then described? ($g = 9{\cdot}8$ metres per sec.)?

15. A body is thrown upwards and reaches a height of $510{\cdot}2$ metres: what was the velocity of projection?

16. What velocity will a falling body acquire in 20 minutes, the acceleration due to gravity being $9{\cdot}8$ metres per second?

17. If the spaces described by a body which has moved from rest under a constant acceleration be in two successive seconds $112\frac{1}{2}$ feet and $137\frac{1}{2}$ feet respectively, what were the spaces described in the two preceding seconds respectively?

18. A body falls from rest for 6 seconds: what space does it describe in the last two seconds?

19. A stone is thrown upward with a velocity of 64 feet per second: find when it is 48 feet above the ground.

20. A moving body is observed to increase its velocity by a velocity of 8 feet per second in every second. How far would it move from rest in 5 seconds?

21. A body known to possess a constant acceleration moves from rest and describes 36 feet in the first 3 seconds. With what velocity will it be moving at the end of the sixth second?

22. A body falls freely from rest through 256 feet. How long will it take to fall through the next 256 ft.?

23. A body falling freely is observed to describe 176 feet in a certain second: how long previously to this second had it been falling?

24. How long will it take a body to fall freely from rest through a vertical height of 192 yards?

25. A, B, C are three points in a vertical line, A being at the top; a body falls from rest from A, and is observed to pass from B to C in 2 seconds: if B is 144 feet above C, how many feet is A above B?

26. Show that when a body falls from rest the space described varies as the square of the time from the beginning of motion.

27. The acceleration due to gravity being 32 when a foot is the unit of length and a second the unit of time, what is the unit of time when the acceleration of gravity is represented by 384, and a yard is the unit of length.

28. When a minute is the unit of time and a yard the unit of length, find the measure of the acceleration with which a train moving from rest acquires in 5 minutes a velocity of 60 miles per hour.

29. Find the measure of the acceleration in the foregoing case when a foot is the unit of length and a second the unit of time.

30. What is the measure of a velocity of 30 miles per hour in feet per second?

31. If an acceleration be expressed by 32 when a foot and a second are the units of length and time respectively, what will be the measure of the acceleration when a yard and a minute are the units?

CHAPTER III.

COMPOSITION AND RESOLUTION OF VELOCITIES AND ACCELERATIONS.

20. *Components and Resultant.*—A body may have at the same time two or more velocities in the same or in different directions, and its actual velocity will be the combined result of these velocities. Thus, suppose a ship drifts with a current, and a man walks along the deck in the direction of the ship's motion, the distance on the earth's surface that he moves over in a second is evidently the sum of the distances moved by the ship and by himself in the second; so that his actual velocity is in this case the *sum* of the two velocities. Similarly when he walks in a direction opposite to the ship's motion, his actual velocity is the difference of the two velocities; and if these velocities be equal his position as regards the earth's surface is unchanged. Again, when a ship acted on by a current sails in a direction making an angle with the current, and a man walks across the deck, his actual velocity is the combined result of three velocities.

The velocities possessed at the same time by any body are called the *Component* velocities, the actual velocity is termed the *Resultant*, and the determination of the resultant when the component velocities are given is called the Composition of Velocities.

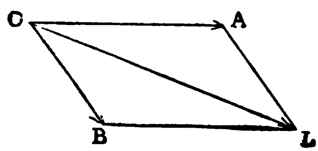
21. *Composition of Velocities in the same straight line.*—If a body have two or more velocities in the same line and in the same direction, the resultant is evidently the *sum* of these velocities. If two velocities are in the same line but in opposite directions their resultant is their difference, and is in the direction of the greater. If a body have any number of velocities in the same line, some in one direction

and some in the other, then calling those in one direction positive and the others negative the resultant is the algebraic sum of the velocities.

22. *Composition of two uniform velocities in different directions.*—If a particle possess at the same time two uniform velocities in directions which are inclined to each other, the resultant velocity is determined by the following proposition:

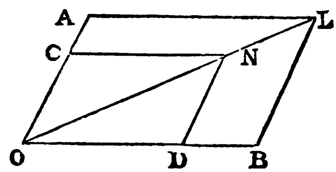
The Parallelogram of Velocities. *If the two uniform velocities possessed by a particle be represented in magnitude and direction by the two adjacent sides of a parallelogram, the diagonal of the parallelogram drawn through the particle will represent the resultant velocity in magnitude and direction.*

Let a particle at O have at the same time two uniform velocities which are represented in magnitude and direction by the lines OA, OB, respectively; then if the parallelogram OALB be formed and the diagonal OL drawn, OL will represent the resultant velocity both in magnitude and direction.



Since the component velocities are both uniform and their directions straight lines, it may be assumed that their resultant will also be a uniform velocity in a straight line. Now, since the particle has two uniform velocities represented by OA and OB respectively, its motion will be the same if we suppose that while the particle moves along the line OA with the velocity represented by OA, the material line OA is moving parallel to itself in the direction OB with the velocity represented by OB. Thus when the particle arrives at the extremity A the line OA will coincide with BL, and the particle will at that instant be at the point L. And as the particle has thus moved from A to L and with a uniform velocity in a straight line, it must have moved along the line OL. Hence OL represents the resultant velocity in magnitude and direction.

[*] 23. The Parallelogram of Velocities being a fundamental proposition in Kinematics, we shall give another proof of the proposition which does not involve the assumption of the preceding article.



Let a particle at O have at the same instant a uniform velocity v along OA, and a uniform velocity v' along OB. Let OA be the distance which the particle would describe in a time t if it moved with the velocity v alone, and let OB be the space it would describe if it moved with the velocity v' alone. Complete the parallelogram OALB, and draw the diagonal OL; then:—

(1) The particle at the end of the time t will be at L;

(2) At any intermediate time it will be found on the line OL;

(3) It will move with uniform velocity along OL.

(1) With the velocity v alone, the particle moving uniformly through v feet per second will in t seconds describe the space vt. Therefore $OA = vt$. Similarly $OB = v't$. Now, since the particle possesses at the same time a velocity v along OA and a velocity v' along OB its motion will be the same if we suppose that the particle moves along OA with a velocity v, while the material line OA carrying the particle along with it moves parallel to itself in the direction of the line BL with the velocity v'. The extremity O of the line OA will consequently move along OB with the velocity v', and in the time t, the line OA will coincide with BL. But in the time t, the particle will have moved along OA to A, and as the point A will then coincide with L, the particle will be at L at the end of the time t.

(2) Let t' be any time intermediate between t and the instant of starting. In the time t' the particle would describe some distance OC with the velocity v alone, and some distance OD with the velocity v' alone. Complete the parallelogram OCND. Then it may be shown as before that in the time

t' the particle will be at N; it is required to prove that N is a point on the line OL.

Since—

$$OA = vt, \qquad OC = vt'$$
$$OB = v't, \qquad OD = v't'$$

Therefore—

$$\frac{OA}{OB} = \frac{vt}{v't} = \frac{v}{v'}$$

and—

$$\frac{OC}{OD} = \frac{vt'}{v't'} = \frac{v}{v'}$$

Therefore, OA : OB :: OC : OD. Hence the parallelograms OALB, OCND are similar, and therefore they are about the same diagonal. Thus N is a point on the line OL.

(3) Since the triangles OAL, OCN are similar—

$$\frac{OL}{ON} = \frac{OA}{OC} = \frac{vt}{vt'} = \frac{t}{t'}$$

That is OL : ON :: t : t'. Hence the spaces passed over by the particle moving along OL are proportional to the times of describing them, and therefore the particle moves with uniform velocity along OL.

Therefore the line OL represents both in magnitude and direction the resultant of the velocities represented by OA and OB.

24. Hence if the two component velocities of a particle be given in magnitude and direction, the resultant velocity is determined geometrically by completing the parallelogram and drawing the diagonal passing through the particle. This diagonal represents the resultant.

25. If the numerical values of the velocities be given and the angle between their directions, the determination of the resultant is the same as the following problem:—

Given the two adjacent sides of a parallelogram and the angle between them to find the diagonal.

26. The solution of this problem requires in general the aid of trigonometry. Some particular cases however may

be solved by elementary geometry. These cases are when the angle between the adjacent sides of the parallelogram has one of the following values:—30°, 45°, 60°, 90°, 120°, 135°, 150°.

Let a particle at O (Figs. 1 and 2) have velocities

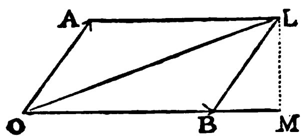


Fig. 1.

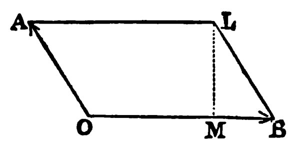


Fig. 2.

represented by OA and OB, which form with each other the angle AOB. Complete the parallelogram and draw the diagonal OL. From L draw LM perpendicular to OB, or OB produced.

Then when AOB is acute, as in Fig. 1, the angle OBL is obtuse, and (Euc. II., 12)—

$$OL^2 = OB^2 + BL^2 + 2OB \cdot BM \qquad (1)$$

And when AOB is obtuse (Fig. 2), then OBL is acute, and (Euc. II., 13)—

$$OL^2 = OB^2 + BL^2 - 2OB \cdot BM \qquad (2)$$

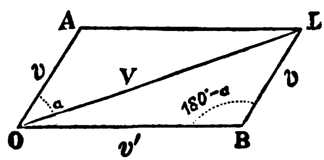
Now when the angle AOB is one of the angles 30°, 60°, 120°, 150°, the triangle LBM is half an equilateral triangle whose side LB is double the shortest side, and as LB = OA, which is given, the sides of the triangle LBM can be calculated, and BM is thus found. And when the angle AOB is either 45° or 135°, the triangle LBM is an isosceles right-angled triangle, and hence BM can be found. Therefore since BM can be determined in all these cases, all the quantities on the right hand side of equations (1) and (2) above are known, and therefore the resultant OL can be calculated.

When the angle AOB is a right angle, then (Euc. I., 47)—

$$OL^2 = LB^2 + OB^2 = OA^2 + OB^2,$$

whence OL can be determined

[*] 27. A general formula may be found by which the resultant of two given velocities whose directions are inclined at a given angle, may be calculated.



Let the particle at O have a velocity v along OA, and a velocity v' along OB, and let OA and OB represent in magnitude and direction these velocities. Complete the parallelogram and draw the diagonal OL. Let OL = V = the resultant velocity; and let $\angle$AOB $= \alpha$. Then OBL $= 180° - \alpha$.

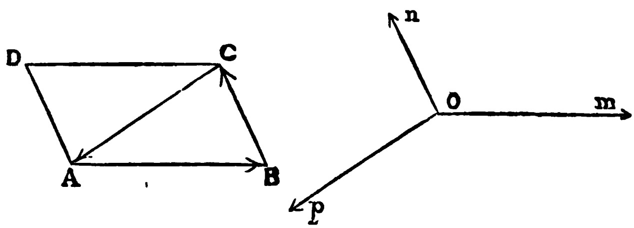
Then by trigonometry—

$$OL^2 = BL^2 + OB^2 - 2BL \cdot OB \cos(180° - \alpha)$$
$$\therefore\ V^2 = v^2 + v'^2 - 2vv' \cos(180° - \alpha)$$
$$\therefore\ V^2 = v^2 + v'^2 + 2vv' \cos \alpha\,;$$

which is the formula required.

28. Triangle of Velocities. *If a particle at rest have given to it at the same instant three velocities which are represented in magnitude and direction by the sides of a triangle taken in order, the particle will remain at rest.*

If a particle possess the three velocities represented by



the sides AB, BC, CA of the triangle ABC taken in order, it will remain at rest.

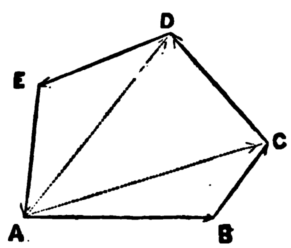
From A draw AD parallel and equal to BC and join DC. Then the velocities AB and BC are represented

by the lines AB and AD. But by the parallelogram of velocities the resultant of the velocities AB and AD is AC. Using this resultant in place of the velocities AB, BC, the particle has two velocities represented by AC and CA, and since these are equal and opposite, the particle remains at rest.

In the foregoing proposition it is not to be understood that the particle tends to move along the lines AB, BC, and CA, but parallel to these directions. If O represents the particle, and if from O we draw Om, On, Op parallel and equal respectively to AB, BC, CA, the lines Om, On, Op will represent the actual paths of the velocities. The expression *taken in order* means that the directions of the velocities are determined by the order of the sides. The first velocity tends from A to B, the second from B to C, and the third from C to A, not from A to C.

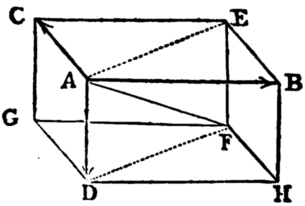
29. Polygon of Velocities. *If a particle at rest have given to it at the same instant velocities which are represented in magnitude and direction by the sides of a polygon taken in order, the particle will remain at rest.*

Let AB, BC, CD, DE, EA represent the velocities: then Art. 22, the velocities AB and BC are equivalent to their resultant AC; AC and CD to AD; and AD and DE to AE. The velocities consequently are equivalent to the two, AE and EA; and as these are equal and opposite, the particle remains at rest.



30. Parallelopiped of Velocities. *If a particle possess at the same time three velocities which are represented in magnitude and position by three adjacent edges of a parallelopiped, the particle will tend to move with a resultant velocity which is represented in magnitude and position by the diagonal of the parallelopiped drawn from the intersection of the three edges.*

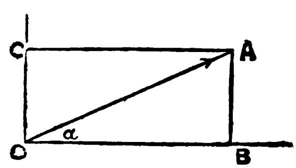
Let AB, AC, AD be the three velocities: then Art. 22, the resultant of AB and AC is AE, and the resultant of AE and AD is AF. Therefore AF is the resultant of the three velocities.



The resultant AF may be calculated when the numerical values of the three velocities are given. For—

$$AF = \sqrt{AE^2 + AD^2} = \sqrt{AB^2 + AC^2 + AD^2}.$$

31. *Resolution of Velocities.*—The Resolution of Velocities is the converse of the Composition of Velocities. It is the finding of the components when the resultant is the given velocity. A given velocity may be resolved into two components by constructing a parallelogram with the line representing the given velocity as diagonal, then two adjacent sides intersecting the diagonal represent the components. In the solution of problems the direction of one of the components is generally determined by the problem, and it is usually found convenient to make the direction of the other component at right angles to the former. The value of the component or *resolved part of the velocity in any direction* can be calculated when the velocity is given, and the angle which its direction makes with the direction in which it is resolved. Thus if OA be a given velocity, and α the angle which its direction makes with the horizontal line, then OB is the resolved part of OA in a horizontal direction, and $OB = OA \cos \alpha$. Similarly $OC =$ the resolved part in a vertical direction $= OA \sin \alpha$.



32. *To determine the resultant of any number of velocities possessed by a particle.*

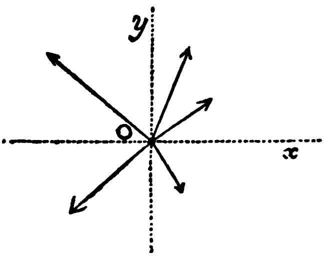
This may be effected by the following methods:—

(*a*) *By repeated applications of the Parallelogram of*

Velocities.—Find, Art. 22, the resultant of two of the velocities, then of this resultant velocity and a third, and so on to the last. The final resultant will be the resultant of all the velocities.

(*b*) *By the Polygon of Velocities.*—Construct a figure whose sides taken in order will represent the magnitude and direction of the velocities; then the line closing the figure and taken in the opposite direction will represent the resultant. Thus, if, in Fig. of Art. 29, AB, BC, CD, DE be lines representing the velocities, then, as was shown in Art. 29, AE represents the resultant velocity.

(*c*) *By resolving the velocities in two directions at right angles, and compounding the results.*



Let O be the particle possessing any number of velocities. Through O draw the axes x and y at right angles to each other. Resolve all the velocities along the horizontal axis, and along the vertical axis. Let the sum of the resolved parts along x be called X, and the sum of the resolved parts along y be Y, then the resultant of X and Y is the resultant of all the velocities. Calling the final resultant V, then since X and Y are at right angles, by the Parallelogram of Velocities—

$$V = \sqrt{X^2 + Y^2}.$$

33. *Composition and Resolution of Accelerations.*—Since Accelerations are the velocities gained in the unit of time, they may also be represented by lines, and may be compounded and resolved precisely in the same way as Velocities. There are thus corresponding propositions in Accelerations to those in Velocities. It will not be necessary to state and prove at length the Parallelogram of Accelerations, the Triangle of Accelerations, &c., since the demonstrations would be exactly similar to those of the corresponding propositions in Velocities.

EXAMPLES.

1. A river whose breadth is 1000 yards runs with a velocity of 3 miles per hour, and a man swims across it, keeping always at right angles to the current, at the rate of 10 feet per second: find (*a*), his resultant velocity; (*b*), how long he will take to cross; (*c*), how far down from the point directly opposite he will land.

$$3 \text{ miles per hour} = 4{\cdot}4 \text{ ft. per sec.} = \text{vel. of stream.}$$

(*a*) $V = \sqrt{v^2 + v'^2} = \sqrt{4{\cdot}4^2 + 10^2} = 10{\cdot}9$ ft. per sec.

(*b*) $\dfrac{1000 \times 3}{10} = 300$ secs. $= 5$ mins.: *or* $\dfrac{\text{actual space described}}{10{\cdot}9} = 5$ mins.

(*c*) $300 \times 4{\cdot}4 = 1320$ feet.

2. A body has two uniform velocities of 20 ft. and 25 ft. per second in directions inclined at an angle of 60°: find the resultant velocity.

$V^2 = v^2 + v'^2 + 2vv' \cos a = 20^2 + 25^2 + 2 \times 20 \times 25 \times \frac{1}{2}$ $\therefore V = 39$ feet per sec nearly.

The resultant velocity may also be found by the method of Art. 26.

3. Find the horizontal and the vertical component of a velocity of 50 ft. per second whose direction makes an angle of 30° with the horizon.

$$\text{Horizontal component} = 50 \cos 30^\circ = 50\frac{\sqrt{3}}{2} = 25\sqrt{3}$$

$$\text{Vertical do.} = 50 \sin 30^\circ = 50 \times \tfrac{1}{2} = 25.$$

4. A particle possesses three uniform velocities of 30 ft., 40 ft., and 50 ft. per second respectively in directions which coincide with three adjacent edges of a parallelopiped: find the space described in 6 seconds.

$$V = \sqrt{30^2 + 40^2 + 50^2} = 70{\cdot}7 \text{ ft.},$$
$$\text{and } 70{\cdot}7 \times 6 = 424{\cdot}2 \text{ ft.}$$

EXERCISES.

1. A body possesses at the same instant velocities of 20 ft., 10 ft., and 8 ft. per second respectively: can it remain at rest?

2. A ship drifted with a current at the rate of 3 miles per hour is impelled by the wind in the opposite direction with a velocity of 180 yards per minute: what is the velocity of the ship relative to the earth's surface.

3. A ship is drifted eastward by a current at the rate of 6 miles per hour and is impelled by the wind southward at the rate of 8 miles per hour: what is the resultant velocity?

4. A body at a point A has given to it at the same instant three uniform velocities, two of which are in the same direction and are 20 ft. and 25 ft. per second respectively, and the other, which is 100 yards per minute, in the opposite direction: where will the body be in 10 seconds?

5. A body possesses a uniform acceleration of 100 feet per second in a direction inclined at an angle of 60° to the horizon: find the components of this acceleration in the horizontal and in the vertical direction.

6. What velocity in a vertical direction is equivalent to a velocity of 500 ft. per second in a direction making with the horizon an angle of 60°?

7. A river is a quarter of a mile wide, and the water flows at the rate of 10 yards per minute. A boat starts from a point in one of the banks and is rowed across the stream at right angles to the current at the rate of 44 yards per minute: where will it land, and how long will it take to cross?

8. A boat is rowed with uniform velocity first up a river and then down it. The river flows at the rate of 2 miles per hour, and the boat is rowed throughout with such force as would give it a velocity of 10 miles per hour in still water: find the actual velocity in each case.

9. A boat rowed at a uniform rate goes $7\frac{1}{2}$ miles in half an hour down the stream, and $6\frac{1}{2}$ miles in half an hour up the stream: find the rate of the stream.

10. A ship sails southward with a velocity of $5\sqrt{3}$ feet per second, and a ball is rolled across the deck towards the west with a velocity of 5 feet per second: find the velocity of the ball relative to the earth's surface, and the angle which its direction makes with the direction of the ship's motion.

11. The resultant of two velocities is 50 feet and its direction makes an angle of 60° with the direction of one of the components. If the component be 25 feet per second, what is the other?

12. Two men starting together run, the one towards the east at the rate of 12 miles per hour, and the other to the north at the rate of 10 miles per hour: how far apart will they be in half an hour?

13. Two roads are inclined to each other at an angle of 60°. Two men travel along the roads each at the rate of 3 miles per hour: how far apart will they be in half an hour?

14. A ship is sailing at the rate of 15 miles per hour, and a sailor climbs the mast at the rate of 10 feet per second: what is his resultant velocity?

15. Find the resultant of two equal velocities of 100 feet per second (*a*), inclined at an angle of 30°; (*b*), at an angle of 120°.

16. A body has a velocity of 25 feet in a minute, and another 15 miles per hour: find the ratio of their velocities.

17. One man rows a mile in 10 minutes, and another 100 yards in 10 seconds: find the ratio of their velocities.

18. A particle has three velocities 8 ft., 8 ft., and 10 ft. respectively whose directions are inclined to each other at angles of 120°: find the resultant velocity.

19. A particle has three velocities of 60 ft. to the east, 30 feet to the north, and 20 feet to the west: find the resultant velocity.

20. A ship is sailing with a velocity of 6 miles an hour, and a sailor climbs the mast at the rate of 10 ft. per sec.: find his resultant velocity.

21. A ship sails southward with the velocity of 15 miles per hour, while it is drifted eastwards by a current at the rate of 3 miles per hour, and a sailor climbs the mast at the rate of 8 feet per second: find his resultant velocity.

22. A ship sailing towards the E. at the rate of 12 miles per hour is drifted by a current to the S.E. at the rate of 3 miles per hour: what distance will the ship sail in 4 hours?

23. A particle has three accelerations of 20, 30, and $20\sqrt{3}$ ft. respectively in directions coinciding with three adjacent edges of a parallelopiped: find the space passed over in 10 seconds.

24. A particle has uniform velocities of 20 metres and 50 metres per second inclined at an angle of $45°$: find the resultant velocity, and the space described in 2 seconds.

25. Accelerations of 50, 100, and $100\sqrt{2}$ ft. per second respectively are in a direction inclined to the horizon at an angle of $45°$: find the value of each in a horizontal direction.

26. A ship sails at the rate of 15 miles per hour, and a man walks across the deck at right angles to the ship's motion at the rate of 10 feet per second: what is his resultant velocity, and how far will he move in 3 secs.?

27. If in the foregoing case the direction in which the man walks makes an angle of $120°$ with the course of the ship, find his velocity.

28. A body possesses accelerations of 10 feet and 12 feet per second respectively in directions inclined at an angle whose cosine is $\cdot 3$: find the resultant acceleration.

29. A body has accelerations of 20 ft. and 30 ft. per second respectively in directions which make an angle whose cosine is $\frac{1}{4}$: find the resultant acceleration, and the space described in 10 seconds.

30. A body possesses simultaneously four equal uniform velocities each of 10 feet per second. The directions of the first and second make an angle of $30°$, those of the second and third $60°$, and those of the third and fourth $30°$: find the resultant velocity, and the space described in 5 seconds.

CHAPTER IV.

DYNAMICS—FORCE.

KINETICS—FORCE AND MOTION.

The First Law of Motion. Composition of Uniform and Accelerated Velocities in the same direction.

34. Up to the present we have considered the motion of bodies without taking into account the cause of the motion. That cause is Force. The relations between force and motion are expressed in Newton's Three Laws of Motion. The first of these laws is given here; the others will be stated and explained in succeeding chapters.

35. NEWTON'S FIRST LAW OF MOTION.—*Every body continues in its state of rest or of uniform motion in a straight line, unless compelled by impressed forces to change that state.*

Body is the name applied to a portion of matter. *Matter* is anything that affects the senses, and which can be acted upon by force. *Force* is whatever changes or tends to change a body's state of rest or of uniform motion in a straight line.

36. The first law of motion expresses that property of matter which is usually denoted by the term *inertia.* By the inertia of matter is meant that it is indifferent either to rest or motion. If at rest matter has no power to put itself in motion, and if in motion it has no tendency to come to rest. The laws of motion are inferences from observation and experiment. They do not admit of direct demonstration, but indirectly their truth may readily be established. That matter *at rest* has no power to put itself in motion, may be assumed as self-evident. That bodies in motion have no tendency to stop, is apparently at variance with observation.

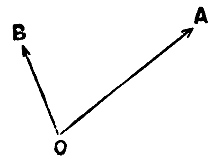
But a little consideration will show that the apparent tendency of bodies in motion to come to rest is really due to forces such as friction and the resistance of the air, which act on the bodies; and that were it not for these forces, the bodies would continue in a state of uniform motion. If these forces are lessened, the motion will last for a longer time. When a body is set in motion on a rough surface, where the friction is great, it soon comes to rest, but if this roughness be diminished the duration of the motion is increased, and the smoother the surface the longer the motion lasts. Hence we infer that if we could remove all opposing forces the state of uniform motion would continue unchanged.

37. Therefore if a body be at rest, or be in uniform motion in a right line, either no force acts on the body, or the forces which act on it equilibrate each other's effects. Matter, so far as we know, does not exist separate from force. What matter is or what force is, we do not know. We can only observe the properties of the one and the effects of the other. One of the chief effects of force is motion, and any cause which changes or tends to change the state of rest or of motion in which a portion of matter may be we call force. If one force only acts on a body, a change in the body's state of rest or motion must ensue. If the body does not change its state of rest or motion, then another force, or forces, must be acting on it, whose effect counterbalances that of the former. A body at rest on a horizontal table is acted on by the attraction of the earth, which tends to move the body vertically downwards, and the body would move in this direction, were it not acted upon by an equal and opposite force, the reaction of the table. When a railway train is in uniform motion on a horizontal road, the reaction of the road equilibrates the weight of the train, and the propelling force acting on the train equilibrates the opposing forces of friction and the resistance of the air. No part of the propelling force is applied to keep up the motion of the train.

38. There are various kinds of forces. Attraction of gravitation, molecular forces, electrical forces, vital forces. At

present we are concerned only with the mechanical effects of forces, and with the methods by which they may be represented and measured.

39. Forces may be *represented* by lines. There are three elements which specify a force, its magnitude, its direction, and its point of application; and all of these may be represented by a line. Thus the line OA represents the magnitude of a force acting at the point O, and in the direction indicated by the arrow head at A. Similarly OB is a force acting from O to B.



40. Forces are *measured* in two ways. (1) A standard force is taken as the measure of all others. This standard force is *weight.* The attraction of gravity is constantly acting upon bodies, and as a consequence all bodies have weight. The weight of a body is the force with which the earth draws the body towards it. The unit of weight usually adopted in this country is the *pound weight.* The weight of a pound is the force with which the earth attracts the quantity of matter in a pound. All other forces may be expressed in terms of this unit. Thus we say a man pulls with a force of 20 lbs., meaning that the muscular effort he exerts is equal to the pull of the earth on the quantity of matter in 20 lbs., and that the one force would therefore exactly counterbalance the other. This method of estimating force is that which is usually employed in Statics.

(2) It follows from the definition of force that its true measure is the *quantity of motion it can produce.* This is the measure of force employed in Kinetics. Quantity of motion will be considered in the next chapter, and this mode of estimating force will then be explained.

41. If a force act on a body for any time, the body, if free to move, will acquire a certain velocity. Then if the force cease its action, and no other force act on the body, it will move in a straight line and with the uniform velocity it

has acquired. If however the force continue its action, a constant acceleration will be added in each unit of time, and the body will therefore move with an accelerated velocity.

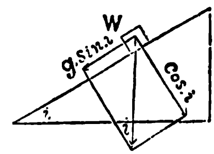
If a body slide down a smooth inclined plane a part of the force of the earth's attraction on the body is neutralized by the reaction of the plane, the acceleration produced is consequently less than g, and the body moves down the plane with an accelerated velocity, the acceleration depending upon the inclination of the plane.

The motion of a falling body is due to the earth's attraction, which is a constant force acting on the body, and which adds an acceleration g in each second to the velocity. A body falls therefore with an accelerated velocity.

42. We may assume—an assumption which will afterwards be justified—that the force of attraction will produce its full effect upon a body independently of any motion that the body may have from the action of any other force. Hence if a stone be thrown in any direction, two velocities are imparted to it, one by the muscular effort, and the other by gravity. The velocity imparted by the former force is uniform, since the muscular action on the stone ceases as soon as the latter leaves the hand; but gravity being a constant force produces an accelerated velocity. The effect of each force is independent of the other, and the position of the stone after any time can be determined by finding its position after the time on the hypothesis that it is acted upon by one of the forces only, and then supposing that it moves from this position under the influence of the second force only for the same time. The principles that have been laid down in Chap. I. will however enable us to deal with the motion of bodies in such cases, in a more concise manner. We proceed to consider some examples of the motion of bodies on inclined planes, and the motion of a body possessing a uniform initial velocity and an accelerated velocity due to gravity.

43. *Motion on Inclined Planes.*—Let W be the weight of a body placed on an inclined plane whose inclination is i.

The weight W would produce an acceleration g vertically downwards if the body were free to move in that direction. Resolving the acceleration g in the direction of the plane, and at right angles to it, the component in the direction of the plane is $g \sin i$, and at right angles to it is $g \cos i$. The latter component is counteracted by the reaction of the plane, and the body will therefore move down the plane with the acceleration $g \sin i$. Thus the relations between space, time, and acceleration, in the case of a body moving down an inclined plane, are determined from the equations of Art. 14, by writing $g \sin i$ for g.



44. The following propositions relate to motion on inclined planes:—

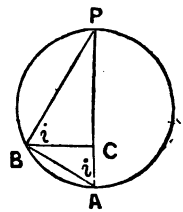
(1) *The velocity acquired by a body in falling down an inclined plane is equal to what it would acquire in falling freely through the height of the plane.*

In the figure of the preceding Art., let l be the length of the plane, and h the height, then $\sin i = \frac{h}{l}$.

By Art. 14, equation (3), the velocity acquired in falling through the height $= \sqrt{2gh}$. By the same equation and the preceding Art., the velocity acquired in falling down the plane—

$$= \sqrt{2g \sin i \, l} = \sqrt{2g \frac{h}{l} l} = \sqrt{2gh}.$$

(2) *If through the highest point of a vertical circle chords be drawn, the time occupied by a body in falling from rest down any chord is constant, and is equal to the time taken by a body in falling freely through the diameter drawn from the same point.*



Let P be the highest point of a vertical circle, PA a vertical diameter, PB any chord. Join BA, and draw BC perpendicular to PA. Let i = angle PBC; then i = angle PAB, and PB = PA sin$i$.

By Art. 14, equation (2), the time of falling down PA = $\sqrt{\frac{2PA}{g}}$.

By Arts. 14 and 43, the time down PB = $\sqrt{\frac{2\,PB}{g\sin i}} = \sqrt{\frac{2\,PA\sin i}{g\sin i}} = \sqrt{\frac{2\,PA}{g}}$.

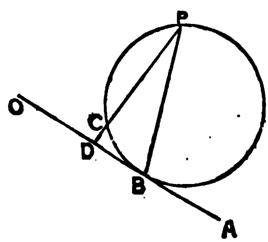
Hence the time down any chord is the same as the time down the diameter.

In the same way it may be shown that the proposition holds for chords drawn through the lowest point.

The proposition is also true for circles whose planes are inclined to the vertical.

45. This proposition enables us to solve a class of problems respecting *lines of quickest descent.* One of these problems is as follows:—

To find the line of quickest descent from a given point to a given line in the same vertical plane.



Let P be the given point, OA the given line. Describe a circle having P its highest point and touching OA in a point B. Then PB is the line of quickest descent. Draw any other line PCD to OA. Then by the preceding proposition the time down PC is equal to the time down PB, therefore the time down PD is greater than that down PB.

46. *Uniform Velocity with Accelerated Velocity.*—We

have now to consider the motion of a body possessing a uniform velocity imparted by some force, and an accelerated velocity produced by gravity. This may be distinguished into two cases.

I. When the uniform and accelerated velocities are in the same line.

II. When they are in different lines.

A stone thrown vertically upwards or downwards is an example of the first. A stone thrown in any direction not vertical is an example of the second.

We suppose in all cases that the motion takes place in a vacuum, so that there is no resistance from the air.

47. I. When a body is projected vertically upwards, and falls back to the point of projection:—

(1) *The time of ascent is equal to the time of descent.*

(2) *The velocity when it returns to the point of projection is equal to the velocity with which it was projected, but is opposite in direction.*

(3) *The velocity at any point in its ascent is equal and opposite to its velocity at the same point in its descent.*

Since the velocity with which the body is projected is opposite in direction to that due to gravity, we use the equations of Art. 14 with the negative signs, where u represents the velocity of projection—

$$v = u - gt \qquad (1)$$
$$s = ut - \tfrac{1}{2}gt^2 \qquad (2)$$
$$v^2 = u^2 - 2gs \qquad (3)$$

When the body reaches its greatest height, it comes to rest for an instant and its velocity is zero. Hence in equation (1) putting $v = 0$ and solving for t, we find the time in which the velocity of the body becomes 0, and therefore the time taken to reach the greatest height. From (1) $0 = u - gt$

$\therefore t = \dfrac{u}{g} =$ time of ascent.

When the body comes back to the point of projection its height above that point is 0. Hence in (2) by putting $s = 0$

and solving for t, we find $0 = ut - \frac{1}{2}gt^2 \therefore t = \frac{2u}{g}$ = whole time of flight. Hence the time of falling $= \frac{2u}{g} - \frac{u}{g} = \frac{u}{g} \therefore$ the time of ascent is equal to the time of descent.

Again the initial velocity with which the body is projected upwards being u, to find the velocity after the time $\frac{2u}{g}$, when the body returns to the point of projection. In equation (1) $v = u - gt$, substituting $\frac{2u}{g}$ for t, then $v = u - g\ \frac{2u}{g} = u - 2u = -u$. Hence the velocities are u and $-u$ respectively, that is they are equal but opposite in direction.

Again any point of the body's ascent may be considered a point of projection, and the velocity at that point as the velocity of projection. Then by the preceding demonstration when the body returns to this point in its descent, its velocity will be equal to the upward velocity at that point but opposite to it in direction.

In equation (1) $v = u - gt$, it will be seen that v continues positive so long as gt is less than u; that is the motion of the body is upwards. When $gt = u$, v becomes 0; that is the body comes to rest. When gt is greater than u, v becomes negative; that is the motion of the body is now downwards.

48. It is to be remembered that the foregoing results are true only on the hypothesis stated in Art. 46, that the bodies are moving in a vacuum. In the air the velocity when the body comes back is less than the velocity of projection, and the time taken in the descent is greater than in the ascent.

When a body is thrown upwards a portion of its motion is destroyed by the resistance of the air, and as a consequence the body does not reach such a height as it would in a vacuum. Hence if it fell even in a vacuum through this height its final velocity would not be equal to that of projection. But during its fall through the air its velocity is retarded by the air, and therefore for both reasons its velocity when it reaches the point of projection is less than its initial velocity.

It follows that its velocity at any point of its descent is less than the velocity at the same point of its ascent. Each indefinitely small portion of the descent must consequently have been described with a less velocity than the same portion of the ascent. The time of describing each portion of the descent is consequently greater than the time of describing the same portion in the ascent. Therefore the whole time of descent is greater than the whole time of ascent.

49. The motion of a body thrown downwards may be deduced from the equations of Art. 14 using the *positive* signs.

EXAMPLES.

1. A stone is projected vertically upwards with a velocity of 160 feet per second: find the greatest height to which it will rise, and the time of reaching that height.

Art. 47. $v^2=u^2-2gs$, where v becomes 0 when the body reaches its greatest height, $\therefore$ $0=u^2-2gs$, and $s=\frac{u^2}{2g}=\frac{160^2}{2\times 32}=400$ ft. = greatest height.

Time of reaching greatest height $=\frac{u}{g}=\frac{160}{32}=5$ seconds.

2. A body is thrown vertically upwards with a velocity of 100 metres per second: when will it return to the point of projection? ($g=9\cdot8$ metres per second.)

Art. 47. $t=\frac{2u}{g}=\frac{2\times 100}{9\cdot 8}=20\cdot4$ secs.

3. A body is projected from the ground with a velocity of 200 feet per second, and half a second afterwards another is projected with a velocity of 400 ft. per second: when will the latter overtake the former?

Let t = the time taken by the first, then $(t-\frac{1}{2})$ secs. = the time taken by second. Substituting these times for t in the general equation $s=ut-\frac{1}{2}gt^2$, we obtain the spaces traversed by each body. These spaces are equal, $\therefore$

$$200\,t-16t^2=400\,(t-\tfrac{1}{2})-16\,(t-\tfrac{1}{2})^2.$$

Solving this equation for t, we find $t=\cdot94$ sec., the time from the starting of the first body.

4. A body is thrown up an inclined plane which rises 1 in 8 with a velocity of 100 ft. per second: how far will it move in 4 seconds?

(Arts. 14 and 43), $s=ut-\frac{1}{2}g\sin i\,t^2=100\times 4-\frac{1}{2}\times 32\times\frac{1}{8}\times 4^2=368$ ft.

5. While a balloon is rising uniformly a stone is let fall from the car which, after 2 seconds, strikes and passes through the glass roof of a building. The stone loses one-half the velocity with which it struck the glass

and it reaches the ground in half a second afterwards: what is the height of the glass roof above the ground? [Balloon's velocity = 40 ft. per sec.]

Art. 14. $v=u-gt=40-32\times 2=-24=24$ downwards.

Velocity after passing through the glass $=\dfrac{24}{2}=12$ ft. per sec. downwards.

$$s=ut+\tfrac{1}{2}gt^2=12\times\tfrac{1}{2}+\tfrac{1}{2}\times 32\times(\tfrac{1}{2})^2=10 \text{ feet.}$$

6. While a balloon is ascending with a uniform velocity of 50 feet per second, the aeronaut throws a stone vertically downwards with a velocity of 30 feet per second. The stone reaches the ground in 10 seconds after its projection: find (*a*), the height of the balloon above the ground at the instant of projection; (*b*), at the instant when the stone reaches the ground; (*c*), the greatest height above the ground to which the stone attains.

The upward velocity of the stone at the instant of projection $=50-30=20$ ft. We employ the equations of Art. 14 with the negative sign.

(*a*) $s=ut-\frac{1}{2}gt^2=20\times 10-16\times 100=200-1600=-1400$. Therefore the stone in 10 seconds will be 1400 feet *below* the point of projection, and hence the height of the balloon at the instant of projection was 1400 ft.

(*b*) $50\times 10+1400=1900$ ft.

(*c*) $v^2=u^2-2gs \therefore 0=u^2-2gs \therefore s=\dfrac{u^2}{2g}=\dfrac{20^2}{2\times 32}=6\frac{1}{4}$ ft., greatest height above point of projection; and $1400+6\frac{1}{4}=$ greatest height above ground $=1406\frac{1}{4}$ ft.

EXERCISES.

1. A body is projected vertically upwards with a velocity $5g$, when will its velocity be $2g$?

Find also when the body will rise to the height $8g$.

2. A stone is projected vertically upwards with a velocity of 96 feet per second: find its height above the point of projection:—(*a*), in 2 seconds; (*b*), in 5 seconds; (*c*), in 6 seconds; (*d*), in 10 seconds; (*e*), in 12 seconds.

3. A stone is projected vertically upwards with a velocity of 64 feet per second from the base of a tower 128 feet high, at the same instant that another is projected downwards from the top of the tower with the same velocity: where will the stones meet?

4. Find in the preceding case where the stones will meet if the stone at the top is allowed merely to fall from the top of the tower.

5. A body is projected vertically with a velocity of 120 feet per second: find (*a*), the greatest height to which it will rise; (*b*), the height in 5 seconds; (*c*), the height in 10 seconds.

6. Find in the foregoing question the time of flight.

7. A body projected vertically from the ground just reaches the top of a tower 256 ft. high: when did it pass the middle point of the tower, and when did it reach the top?

8. In the foregoing question find the velocity with which the body is moving when it reaches the middle point of the tower going up and coming down.

9. A body falls from rest down an inclined plane which rises 1 in 10: find the velocity after 4 seconds.

10. Find the space described in 5 seconds by a body which falls from rest down an inclined plane whose inclination is 30°.

11. An inclined plane rises 1 in 16, and a body is thrown up the plane with a velocity of 128 feet per second, and just reaches the top when it comes to rest: what is the length of the plane?

12. A body falls from rest down an incline whose length is 128 feet and inclination 30°: find the time.

13. A body is thrown up an inclined plane which rises 1 in 10 with a velocity of 50 feet per second: (*a*), what is the velocity after 5 seconds? (*b*) When will the body stop?

14. A plane rises 1 in 64 and is 225 ft. long: with what velocity must a body be projected up the plane so as just to reach the top?

15. A body is projected down a plane whose inclination is 60° with a velocity of 50 ft. per second: find the velocity and the space described in 3″.

16. A railway carriage runs from rest down an incline of 1 in 100: find the space described in 5 minutes.

17. In the preceding case, find what space would be described in a minute if the carriage started with a velocity of 15 miles per hour.

18. A stone falls from the top of a precipice, and 2 seconds afterwards another falls from a point 192 feet lower down: when and where will the first stone overtake the second?

19. A stone is thrown downwards with a velocity of 20 feet per second, and 2 seconds afterwards another is projected after it with a velocity of 100 feet per second: when will it overtake the first?

20. A stone is thrown upwards with a velocity of 120 feet per second, and 2 seconds afterwards another is projected after it with the velocity of 200 ft. per second: when will the stones be together?

21. A chord drawn from the highest point of a vertical circle makes an angle of 45° with the vertical diameter, and is 32 feet long: find the time taken by a body to fall from rest down the chord.

22. Find the time of running down a chord 24 feet long drawn from the highest point of a vertical circle at an angle of 30° with the vertical diameter.

23. A body receives in one direction a uniform velocity of 10 feet per second, and in a direction inclined at an angle of 60° to the former an acceleration of 2 feet per second: what will its velocity be at the end of 5 seconds?

24. A body is projected horizontally with a velocity of 100 feet per second and receives an acceleration of 32 ft. per second vertically downwards: what will be its velocity in 5 seconds, and what will be its distance from the point of projection?

25. A body is projected at an angle of 30° to the horizon with a velocity of 48 feet per second, and is acted on by gravity: what will be its velocity at the end of 3 seconds?

26. A body receives a uniform velocity in one direction with which it would describe a space of 512 feet in 4 seconds, and it receives an accelera-

tion in a direction inclined at an angle of 120° to the former with which it would describe in the same time 256 feet: what will be its velocity at the end of the 4 seconds?

27. While a balloon is ascending with a uniform velocity of 50 feet per second, the aeronaut throws a stone vertically upwards with a velocity of 30 feet per second. The stone reaches the ground in 10 seconds after its projection: what was the height of the balloon at the instant of projection?

28. If the balloon in the foregoing question continues ascending uniformly: find (*a*), the height of the balloon above the ground when the stone reaches it; (*b*), the greatest height above the ground which the stone attains.

29. While a balloon is descending uniformly with a velocity of 100 ft. per second, the aeronaut throws a stone vertically upwards with a velocity of 100 ft. per second. The stone reaches the ground in 5 seconds: what was the height above the ground at the instant of projection?

30. A stone is projected horizontally from the top of a tower with a velocity of 100 feet per second, and reaches the ground in $2\frac{1}{2}$ seconds: find (1), the height of the tower; (2), the distance from the point where it reaches the ground to the base of the tower; (3), the distance from the same point to the top of the tower.

31. A man standing up in a boat is likely to fall if the boat suddenly move on; and if standing while the boat is in motion he is likely to fall if the boat suddenly strike the bank: state in each case the direction the man falls, and the reason.

32. While a train is in motion at the rate of 30 miles per hour, a screw in the roof of one of the carriages becomes detached and falls for one second when it strikes the floor of the carriage: (*a*), what path will it appear to a person in the carriage to describe? (*b*) How far does it move in a horizontal direction while falling? (*c*) How far in a vertical direction?

33. While a circus horse is galloping with uniform velocity the rider throws a ball upwards, and it returns again to his hand: in what direction was it thrown?

34. While a railway train is moving with a uniform velocity of 50 feet per second a stone is thrown vertically upwards with a velocity of 80 feet per second from an open carriage in the train. Neglecting the resistance of the air: find (*a*), the greatest height it reaches; (*b*), the time of flight; (*c*), the horizontal range.

35. A balloon ascends from the ground with a uniform velocity of 16 feet per second, and after 10 seconds a stone is let fall from the car: when will it reach the ground?

36. A lift ascends from the ground with a uniform acceleration of 16 feet per second, and after 2 seconds a stone is let fall from it: when will it reach the ground?

37. A lift ascends from the ground with a uniform acceleration of 16 feet per second, and after 10 seconds a stone is let fall from it: when will it reach the ground?

38. A ball is allowed to fall to the ground from a certain height, and at the same instant another ball is thrown upwards with just sufficient velocity to carry it to the height from which the first one fell: show when and where the two balls will pass each other.

39. The intensity of gravity at the surface of the planet Jupiter being about 2·6 times as great as it is at the surface of the earth, find the time which a heavy body would occupy in falling from a height of 167 feet to the surface of the planet Jupiter.

40. A heavy body on a level plane has simultaneously communicated to it an upward vertical velocity of 48 feet per second and a horizontal velocity of 25 feet per second: find (*a*), its greatest height; (*b*), its range; (*c*), whole time of flight.

41. A body is projected up an inclined plane whose inclination is 30° with a velocity of 16 feet per second: (1), how far will it move before coming to rest; (2), how far will it be from the starting point after 5 seconds from the beginning of motion?

42. A balloon has been ascending vertically at a uniform rate for 4·5 seconds, and a stone let fall from it reaches the ground in 7 seconds: find the velocity of the balloon and the height from which the stone has been let fall.

43. A tower is 100 feet high and a stone is projected upwards from the base with a velocity which just carries it to the top: when will it be within 20 feet of the top?

44. If a stone be thrown vertically upwards with a velocity of 200 feet per second: how long will it be before it comes back to the point of projection?

45. If a body is thrown vertically upwards and moves for 6 seconds before it stops, what was the velocity with which it was thrown?

46. A stone is thrown vertically upwards in a vacuum, and after reaching a certain height it returns to the point of projection: prove that it took the same time in its ascent as in its descent.

47. At the instant when a balloon is 400 yards above the ground and ascending with a velocity of 16 feet per second, a stone is dropped from the car: (*a*), where will the stone be at the end of one second, and (*b*), when will it reach the ground?

48. A body is gaining every minute a velocity of 15 miles per minute: what is the measure of the acceleration when the units are the foot and the second respectively?

49. If 32 be the measure of the acceleration produced by gravity when a foot and a second are the units of length and time respectively, what will be the measure when the unit of length is 5 yards and the unit of time 5 minutes?

[*] CHAPTER V.

PROJECTILES.

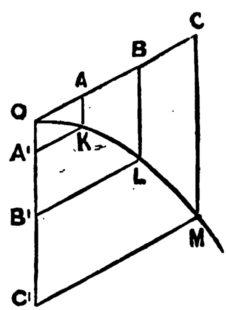
Composition of Uniform and Accelerated Velocities in different directions.

50. We shall consider in this chapter case II. of Art. 46.

51. It has been shown in Art. 22 that when a body possesses at the same time two *uniform* velocities it will move with uniform velocity along the diagonal of the parallelogram having for adjacent sides lines representing the two component velocities.

If a body possess a *uniform* velocity in one direction, and an *accelerated* velocity in another, the *position* of the body at the end of any time will be determined in the same way as in Art. 22, but the body will not move in a straight line along the diagonal.

52. Let a body at O possess a uniform velocity along OC and an accelerated velocity along OC′; let OA, OB, OC, &c., be the paths it would describe in 1 second, 2 seconds, 3 seconds, &c., respectively, if moving with the uniform velocity alone, and OA′, OB′, OC′ be the paths in the same times, if moving with the accelerated velocity alone. Then as in Art. 22 it may be shown that the body will be at K in one second, at L in two seconds, at M in three seconds, &c. Thus



the position of the body possessing simultaneously the two velocities will be the same at the end of the first second, as

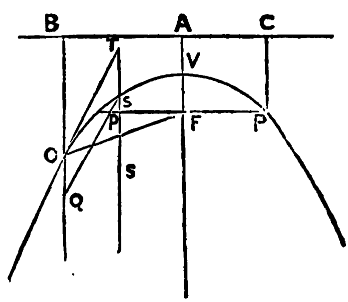
if it had first moved from O to A, and then from A to K. In the same way the position of the body at intermediate times may be determined.

Since the distances OA′, A′B′, B′C′ are unequal, and OA, AB, BC equal it is evident that the points K, L, M do not lie upon the diagonal joining OM. If an indefinite number of such parallelograms be constructed so as to determine the positions of the body at intermediate times the angular points of these parallelograms remote from O would form a curve OKLM, the nature of which will depend upon the kind of acceleration along OC′.

53. If a body be projected in any direction which is not vertical a uniform velocity is given to it by the force of projection, and an accelerated velocity by gravity. Such a body is called a *projectile*, and we have now to consider the motion of projectiles. The body is supposed to be moving in a *vacuum*, so that there is no resistance from the air, and during the motion gravity is the only force acting on the body.

54. Before proceeding to consider the nature of the path of a projectile, the student should be acquainted with the following definitions and properties of the *parabola:*

Def. *If a fixed straight line and a fixed point be taken, and if a curve be drawn so that any point in it is equidistant from the fixed point and the fixed straight line, the curve is called a parabola.*



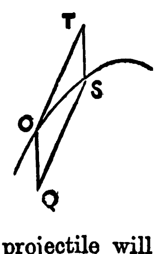
Let BAC be a fixed line and F a fixed point, and let the curve OSVP be such that any point in it is equidistant from F and from BC, then the curve is a parabola. Take any point O and draw OB perpendicular to BC, and join OF then OB = OF. Similarly PC = PF and VA = VF. The fixed line BC is called the *directrix.*

and the point F the *focus*. The straight line AF through F and perpendicular to the directrix is the *axis* of the parabola. The point V where the axis intersects the curve is the *vertex*. The line PP drawn through the focus at right angles to the axis and terminated by the curve is the *latus rectum*. The latus rectum $PP = 2FP = 2PC = 4FV = 4AV$. The line drawn from any point in the curve touching the curve is a *tangent*. A line drawn parallel to the axis from any point in the curve is a *diameter*. In the figure OT is a tangent, and SS a diameter. A line SQ drawn from any point S parallel to a tangent OT and terminated by the diameter through O is termed an *ordinate*, and the portion of the diameter OQ is called an *abscissa*.

It is proved in Conic Sections that the ratio of the square of the ordinate to the abscissa is constant, and is equal to four times the distance of the point O from the directrix or from the focus. Hence wherever the points O and S are taken the ratio $\frac{SQ^2}{OQ}$ is constant and is equal to 4OF. This is a characteristic property of the parabola, and if this property is found in any curve, we infer that the curve is a parabola.

55. We proceed to demonstrate some propositions regarding the motion of projectiles :—

The path of a projectile is a parabola. Let a body be projected in a vacuum from O in the direction OT, which is not vertical, and with the velocity u. Let OT be the line which the body would describe in the time t if it were not acted upon by gravity. From T draw TS vertical and equal to the space which the body would describe in the time t under the influence of gravity. Then, Art. 22, the projectile will be at S after the time t; and by Art. 52, it will move in some curve from O to S during this time. Complete the parallelogram $QS = OT = ut$, and $OQ = TS = \frac{1}{2}gt^2$. Therefore $QS^2 =$



$\frac{2u^2}{g}$ OQ. Therefore QS^2 is equal to OQ multiplied by a constant, and hence the square of the ordinate varies as the abscissa. Therefore the curve OS is a parabola whose axis is vertical; and the distance of O from the focus, and from the directrix is one-fourth of $\frac{2u^2}{g} = \frac{u^2}{2g}$.

56. In the following propositions on projectiles we shall suppose that in every case the initial velocity of projection is u, and that the direction of projection makes an angle α with the horizontal plane. It follows from Art. 31 that resolving this velocity horizontally and vertically, the horizontal component is $u \cos\alpha$, and the vertical $u \sin\alpha$. We may regard the body as moving with these two velocities, and we assume, as in Art. 42, that each velocity is independent of the other. So far as *vertical* effects are concerned, we may regard the body as moving with the velocity $u \sin\alpha$ only, and so far as regards *horizontal* effects, with the velocity $u \cos\alpha$ only. The vertical velocity of a projectile constantly changes owing to the opposite acceleration produced by gravity. There being no opposing force in a horizontal direction, the horizontal velocity remains the same throughout the whole flight.

57. *The velocity of a projectile at any point of its path is equal to what it would acquire if it fell from rest from the directrix to that point.*

From the point of projection where the velocity is u, the distance of the directrix is $\frac{u^2}{2g}$ (Art. 55); and by Art. 14, equation 3, if a body fall through the space $\frac{u^2}{2g}$ it will acquire the velocity u. Now any point of the path of a projectile may be considered the point of projection, and the velocity there, the velocity of projection. Thus if at any other point the velocity be v, the distance of the directrix is $\frac{v^2}{2g}$, and, Art. 14, a body falling through this height will acquire the

velocity v. Hence the velocity at any point is equal to what would be acquired in falling from rest to that point from the directrix.

58. *To find the latus rectum of the parabola described by a projectile.*

By Art. 54 the latus rectum is equal to four times the distance of the vertex from the directrix. At the vertex the vertical velocity is zero, and therefore the whole velocity is the same as the horizontal component and is therefore $u \cos\alpha$. The height from which a body must fall to acquire this velocity is, Art. 14, $\frac{u^2 \cos^2\alpha}{2g}$ and by last Art. this is the distance from the vertex to the directrix. Four times this distance or $\frac{2u^2 \cos^2\alpha}{g}$ is (Art. 54) the latus rectum.

59. *To find the time in which a projectile reaches its greatest height.*

The velocity of projection being u, and α the angle made with the horizon, the vertical velocity is $u \sin\alpha$. Let t = time taken to reach the greatest height. In the general equation 4 of Art. 14 substituting for u the vertical velocity $u \sin\alpha$, we obtain—

$$v = u \sin\alpha - gt.$$

When the body reaches its greatest height v becomes 0, and the equation becomes—

$$0 = u \sin\alpha - gt$$

$$\therefore t = \frac{u \sin\alpha}{g}$$

60. *To find the greatest height.*

From Art 14 $v^2 = u^2 - 2gs$. Substituting for u the vertical velocity $u \sin\alpha$, and since v becomes 0 when the body reaches its greatest height, therefore—

$$0 = u^2 \sin^2\alpha - 2gs$$

$$\therefore s = \frac{u^2 \sin^2\alpha}{2g}.$$

61. *To find the whole time of flight.*

The time of flight is the time that elapses from projection till the projectile returns to the horizontal plane passing through the point of projection. Let t = time of flight. From Art. 14, $s = ut - \frac{1}{2}gt^2$, where s denotes the height above the horizontal plane. This height is 0 at the time t, therefore substituting the vertical velocity $u \sin\alpha$ for u, we obtain—

$$0 = u \sin\alpha \; t - \tfrac{1}{2}gt^2$$

$$\therefore t = \frac{2u \sin\alpha}{g}.$$

62. *The time of ascent of a projectile is equal to the time of descent.* By last Art. the whole time of flight is $\frac{2u \sin\alpha}{g}$, and the time of ascent is, by Art. 59, $\frac{u \sin\alpha}{g}$; therefore the time of descent is $\frac{2u \sin\alpha}{g} - \frac{u \sin\alpha}{g} = \frac{u \sin\alpha}{g}$.

63. *The vertical velocity on returning to the horizontal plane is equal to the vertical component of projection, but is opposite in direction.*

By Art. 14 the vertical velocity in the time t is given by the equation $v = u - gt$. Substituting for u, $u \sin\alpha$, and for t, the time of flight as found by Art. 61—

$$v = u \sin\alpha - g \frac{2u \sin\alpha}{g} = -u \sin\alpha.$$

Therefore the vertical velocity on returning to the horizontal plane is equal to the vertical velocity of projection, but is opposite in direction.

Since any point of the path of a projectile may be taken as the point of projection, and its velocity there as the velocity of projection, it follows that the vertical velocities at any two points of the path of a projectile which are in the same horizontal plane are equal but opposite in direction. And as

the horizontal velocity remains the same throughout the whole flight both in magnitude and direction, therefore the whole velocities at these two points are also equal but oppositely inclined to the plane and making equal angles with it.

64. *To find the range on the horizontal plane.*—By Art. 61 the time of flight is $\frac{2u \sin\alpha}{g}$, and the uniform horizontal velocity is $u \cos\alpha$, therefore, Art. 6, the horizontal space described, or the range, is—

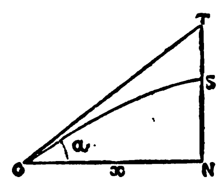
$$\frac{2u \sin\alpha}{g} \times u \cos\alpha = \frac{2}{g} u \sin\alpha \, u \cos\alpha = \frac{u^2 \, 2\sin\alpha \cos\alpha}{g} = \frac{u^2 \sin 2\alpha}{g}.$$

65. With a given velocity of projection u, the range will be the greatest when $\sin 2\alpha$ has its greatest value, that is when $2\alpha = 90°$, and $\alpha = 45°$.

Again calling the range r, then $r = \frac{u^2 \sin 2\alpha}{g}$, and $u^2 = \frac{rg}{\sin 2\alpha}$. Hence u has its least value when $\sin 2\alpha$ is greatest, that is when $\alpha = 45°$.

Therefore with a given velocity the greatest range is obtained when the angle of projection is made $45°$; and with a given range the least velocity is required when the angle of projection is $45°$.

66. *To find the equation to the curve described by a projectile.*



Let OS be a portion of the path of a projectile, S any point of the curve, SN a vertical through S, meeting the horizontal line ON drawn through the point of projection; then the equation to the curve is the expression of the relation between the co-ordinates SN and ON. Let $SN = y$, and $ON = x$. Let u be the velocity of projection, α the angle of projection, and t the time at which the projectile reaches S, and let the vertical through S meet the direction of projection at T. Then since the horizontal velocity is uniform and equal to $u \cos\alpha$, the

horizontal space described in the time t is $u \cos\alpha\, t$ (Art. 6), therefore—

$$x = u \cos\alpha\, t \qquad (1)$$

Again from Art. 14 the vertical space in the time t with the vertical velocity $u \sin\alpha$ is $u \sin\alpha\, t - \frac{1}{2}gt^2$, therefore—

$$y = u \sin\alpha\, t - \tfrac{1}{2}gt^2 \qquad (2)$$

Finding the value of t in (1) and substituting it in (2) we obtain—

$$y = x \tan\alpha - \frac{gx^2}{2u^2 \cos^2\alpha} \qquad (3)$$

These equations will be found very useful in the solution of problems. The last enables us to solve the following problem:—

67. *To find the angle at which a body must be projected with a given velocity so as to hit a given point.*

Let the point be S. Then since S is given x and y are known, and in equation (3) all the quantities are known except α, which is the angle of projection. Since $\frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$ this equation may be written—

$$y = x \tan\alpha - \frac{gx^2}{2u^2}(1 + \tan^2\alpha).$$

Solving this equation for $\tan\alpha$, we find the angle of projection.

When the two roots of this quadratic equation are both real and unequal there are two angles of projection, and two solutions of the problem; when both roots are real and equal, there is only one solution; and when both are imaginary there is no solution and the problem is impossible.

68. The results of Art. 47 should be compared with those of the preceding Arts. Some of these results are placed side by side in the following table. It will be seen that so far as vertical effects are concerned the results of the preceding Arts. can be obtained from those of Art. 47 by writing $u \sin\alpha$ for u.

	Body projected vertically with velocity u.	Body projected at an angle a with horizontal plane with velocity u.
Vertical component of velocity, . . .	u	$u\sin a$
Horizontal component, .	o	$u\cos a$
Vertical velocity in time t,	$u-gt$	$u\sin a-gt$
Equation connecting vertical velocity and space, .	$v^2=u^2-2gs$	$v^2=u^2\sin^2 a-2gs$
Time of greatest height, .	$\frac{u}{g}$	$\frac{u\sin a}{g}$
Greatest height, . .	$\frac{u^2}{2g}$	$\frac{u^2\sin^2 a}{2g}$
Whole time of flight, .	$\frac{2u}{g}$	$\frac{2u\sin a}{g}$
Horizontal range, . .	o	$\frac{u^2\sin 2a}{g}$

69. The foregoing investigation of the motion of projectiles in a vacuum possesses only a theoretical interest, and has no practical value. Owing to the resistance of the air, the path of a body projected with any considerable velocity is not a parabola, and the results observed differ very widely from those obtained by calculation. The theory affords however a valuable illustration of the principles that have been explained, and should therefore receive the careful attention of the student.

EXAMPLES.

1. A body is projected with a velocity of 120 feet per second at an angle of 60° with the horizon: find (1), the greatest height; (2), the time of flight; (3), the range.

(1) By Art. 60, the greatest height $=\frac{u^2\sin^2 a}{2g}=\frac{120^2\times\frac{3}{4}}{2\times 32}=168{\cdot}75$ feet.

(2) By Art. 61, the time of flight $=\frac{2u\sin\alpha}{g}=\frac{2\times 120\times\frac{\sqrt{3}}{2}}{32}=6\cdot49$ secs

(3) Art. 64, the range $=\frac{u^2}{g}\sin 2\alpha=\frac{120^2\times\frac{\sqrt{3}}{2}}{32}=225\sqrt{3}$ ft.

2. A body is projected with a velocity of $8g$ feet per second at an inclination of 60° to the horizontal plane: what is the range?

Art. 64, range $=\frac{u^2}{g}\sin 2\alpha=\frac{(8g)^2}{g}\sin 120^\circ=64g\times\frac{\sqrt{3}}{2}=1024\sqrt{3}$ ft.

Exercises.

1. While a railway train is moving uniformly with a velocity of 50 feet per second, a body is projected from the train in the direction in which it is moving with a velocity of 32 feet per second and at an angle of 60° with the horizon: find (*a*), the greatest height to which the projectile will rise (*b*), the time of flight; (*c*), the horizontal range.

2. A body is projected at an angle of 30° to the horizon with a velocity of 120 feet: find its height at the end of 2 seconds.

3. In the foregoing question if the projectile hit a vertical wall in 3 seconds, how far is the wall away?

4. Three bodies are projected horizontally from the top of a tower 64 feet high with velocities of 80 ft., 100 ft., and 120 ft. respectively: find the time in each case when the body comes to the ground, and the horizontal distance traversed by each.

5. A body is projected at an angle of 45° to the horizon, and in 4 secs. it just passes horizontally over the top of a wall: what was the velocity of projection?

6. A cannon ball is fired horizontally with a velocity of 1500 ft. per second from the top of a hill, and strikes the level plain 5 seconds afterwards: what is the height of the hill above the plain?

7. From the foregoing question find the horizontal distance traversed by the cannon ball.

8. A body is projected at an angle to the horizon whose sine is $\frac{1}{8}$, with a velocity of 256 feet per second: find the greatest height to which it will rise, and the time of flight.

9. A body is projected at an angle of 30° to the horizon with a velocity of 256 feet per second, and at the same instant another is projected vertically. Both reach the ground at the same time: what was the velocity of the latter?

10. A body is projected with a velocity of 320 ft. per second at an inclination of 75° to the horizon: what is the range?

11. What would the range have been in last question if the angle of projection had been $45°$?

12. Three bodies are projected horizontally from the top of a tower 64 feet high with velocities of 100, 200, and 300 ft. respectively: when will they reach the horizontal plane?

13. A body is projected with a velocity u at an angle α to the horizon: how long does it take to reach the further end of the latus rectum of the parabola which it describes?

14. The velocities of a projectile at two points of its path are respectively 96 feet and 64 feet: find the distance of the focus of the parabolic path from the first point, and the distance of the directrix from the second.

15. Four bodies are projected with equal velocities of 96 feet per second, and at angles with the horizon of $30°$, $45°$, $75°$, and $90°$ respectively: find the range of each.

16. The range of a projectile is 512 feet, and the angle of projection 45: what was the velocity of projection?

17. A body is projected at an angle with the horizon whose sine is $\frac{1}{4}$, and just passes horizontally over a wall 64 ft. high: what was the velocity of projection?

CHAPTER VI.

SECOND LAW OF MOTION.

Relations between Force, Mass, and Acceleration.

70. We have seen that when a force acts on a body it tends to move it, and if the body be free to move, it will be set in motion in the direction in which the force acts. The velocity communicated to a body by a given force in a given time will however vary with the quantity of matter in the body. The quantity of matter in a body is called its *Mass;* and experiment shows that the following relations subsist between the mass of a body, the force acting upon it, and the acceleration produced :—

(1) A constant force acting on the same mass produces a constant acceleration.

(2) If different forces act on the same mass the accelerations are proportional to the forces.

(3) If the same force acts on different masses, the accelerations are inversely proportional to the masses.

To these we add the statement made above.

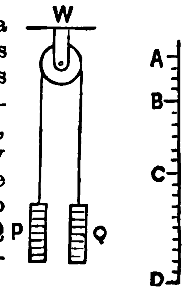
(4) When a force produces motion in a body the motion takes place in the direction in which the force acts.

All these statements are included in Newton's SECOND LAW OF MOTION, which is as follows :—

The change in the quantity of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

71. The foregoing relations between force, mass, and

acceleration may be verified approximately by experiment in several ways. The following is one of the most satisfactory :—



A fine silk string is passed round a smooth pulley W, which turns freely on its axis, and to the ends of the string weights P and Q are attached. P and Q are composed of a number of thin circular discs, all equal in weight, which can be readily removed or replaced at pleasure; and the mass of the pulley and the string is so small compared with that of P and Q that it may be neglected in the experiment.

If P and Q be equal, the system will remain at rest, since the tension acting upwards on each weight is exactly balanced by the weight acting downwards. If one of the weights P be made greater than Q, the system will be set in motion, P downwards and Q upwards. The force causing motion is $P - Q$, the difference of the weights P and Q, and the mass moved is the quantity of matter in $P + Q$.

If now we remove an equal number of discs from P and Q, or add an equal number to each, the mass moved becomes changed while the force causing motion, $P - Q$, remains the same. And by taking some of the discs from Q and placing them on P, or conversely, we change the force $P - Q$ which produces the motion while the mass moved remains unaltered. We have thus a ready means of varying the force while the mass remains constant, or varying the mass while the force remains constant.

A vertical scale AD is fixed near the instrument, which may be used as follows to verify the statements of Art. 70 :—

72. One of the weights P being made greater than the other, P is raised to some point in the scale, and when the system is at rest, the support under P is withdrawn and the spaces described by it in successive seconds are observed. Suppose at the end of the first second P arrives at B, at the end of the second second at C, at the third at D, and so on.

The spaces AB, BC, CD, &c., are then measured, and it is found that if the distance from A to B be called d, BC is equal to $3d$, CD to $5d$, DE to $7d$, &c. Hence a constant increase of $2d$ is added to the velocity in each second, and thus a constant acceleration is produced by the same force $P - Q$ acting on the same mass. If now we vary the weights P and Q and repeat the experiment, a new acceleration is obtained which is also constant. Hence we infer that the *same force acting on the same mass produces a constant acceleration.*

Again take weights P and Q, and determine the acceleration as before with the force $P - Q$. Now remove some of the discs from one weight and place them on the other. The force which produces motion is different, while the mass moved remains the same. Determining the acceleration in the latter case and comparing it with the acceleration due to the force $P - Q$, it is found that *the accelerations are proportional to the forces.*

If once more we determine the acceleration with any weights P and Q, and that we then remove equal weights from P and Q, or add equal weights to them, and find the new acceleration. The force producing motion is in both cases $P - Q$, while the masses moved are different; and it is found that the *accelerations observed are inversely proportional to the masses.*

The fourth of the statements in Art. 70 is also verified by the preceding experiments, since in all cases the forces acting on the body produce motion or tend to produce it in the direction in which they act. This is true whether the body be at rest or in motion and whatever be the number of forces acting on it. Each force tends to move the body in its own direction, and if several motions be given to the body its resulting motion is the combined effect of all the forces. Thus if in figure of Art. 22 two forces acting on a body at O in the directions OA and OB produce velocities in one second represented by OA and OB respectively, then in one second the body will cross the line AL and also the line BL, and therefore at the end of a second must be found at the

intersection L of these lines. Thus each force has its full effect independently of the other, and the same is true in the case of any number of forces acting on a body. The assumption therefore of Art. 42 is a corollary from the Second Law of Motion.

73. *Atwood's Machine.*—The arrangement described in Art. 71 is the essential portion of an instrument called, from the inventor, Atwood's Machine. As usually constructed the axle of the pulley rests on friction wheels, the scale is furnished with stops, and a pendulum beating seconds is attached. Owing to the slight friction the motion of the weights continues for a considerable time, and the distances described can be determined with great accuracy.

This instrument may be employed to verify the laws of falling bodies which have already been demonstrated. The spaces described by a body falling freely are so large that it is almost impossible to test the truth of the laws stated in Chap I. by the observation of a body falling freely under the influence of gravity. With Atwood's Machine we can however make the acceleration as small as we please, and we can then readily observe the spaces described. In the case of a body falling freely the force producing motion is the weight of the body, the mass moved is the quantity of matter in the body, and the acceleration is g. With Atwood's Machine the force $P - Q$ producing motion can be diminished to any amount, while the mass moved remains unaltered, and thus the acceleration may be made as small as we wish. From Art. 14 the laws of the motion will however be the same as those for bodies falling freely. Hence if with Atwood's Machine the system be started from rest, and the spaces described in 1, 2, 3, &c., seconds be determined, it will be found that these *spaces are proportional to the squares of the natural numbers.* If again the spaces described in successive seconds be measured, it will be found that these *are proportional to the odd numbers* 1, 3, 5, 7, &c. And in a similar way the other theorems of Chap. I. may be verified.

74. The Inclined Plane can be used for the same purpose as Atwood's Machine. Galileo first employed it to illustrate

the laws of falling bodies. As explained in Art. 43 we can diminish to any extent the acceleration on the inclined plane, and we are thus enabled easily to observe the spaces described by a body moving down the plane. The inclined plane may also be used to verify the statements of Art. 70.

75. Propositions 2 and 3 of Art. 70 express the relations between the mass of a body, the force acting upon it, and the acceleration produced. If F denote the force, m the mass, and f the acceleration, then according to these propositions, f varies directly as F if m be constant, and varies inversely as m if F be constant. Both of these statements are expressed by the algebraic formula—

$$f \text{ varies as } \frac{F}{m}.$$

When one quantity varies as another, the first is equal to the second multiplied by some constant. This constant can be made unity by a proper choice of units. The foregoing expression can therefore be written—

$$f = \frac{F}{m}, \text{ or } F = mf.$$

This equation expresses in a concise form the relations between force, mass, and acceleration, and is of great use in the solution of problems.

76. When a body falls freely under the influence of gravity, the force acting upon the body is its weight, which is the amount of the earth's pull upon the quantity of matter in the body. Calling W the weight, m the mass, and g the acceleration due to gravity, the foregoing equation becomes—

$$g = \frac{W}{m}, \text{ or } W = mg.$$

77. If we assume that the weights of bodies are proportional to their masses, it follows that if we increase or decrease the weight of a body the mass is increased or decreased in the same ratio, and therefore by the foregoing equation the value of g remains unaltered. A light body should therefore fall with the same velocity as a heavy one, because although the force causing motion is less in the light than

in the heavy body, the mass moved is also less and in the same ratio, and hence the acceleration would be the same in both cases. It is found by actual experiment that in a vacuum, where there is no resistance from the air, all bodies light and heavy fall with the same velocity. The assumption is therefore justified, that independently of the kinds of matter in bodies, their weights are proportional to their quantities of matter, and that the earth attracts all kinds of matter alike.

78. Since the weight of a body acting on the quantity of matter in the body communicates the acceleration g, we have thus a standard force and an acceleration with which others may be compared. If for instance W be the weight of the body, and it is required to find the acceleration that a force F would produce if applied to the body, then since the weight W is a force which would communicate to the body an acceleration g, and since when the mass is the same the accelerations are proportional to the forces, therefore F will communicate an acceleration f, which is obtained from the proportion—

$$\mathrm{W} : \mathrm{F} :: g : f \therefore f = \frac{\mathrm{F}}{\mathrm{W}} g.$$

79. *Units of Force and Mass.*—In Art. 75 by writing the variation as an equation we are restricted in the units of force and mass we may employ, one of them being determined by the equation when the other is chosen. The unit of force is that force which will produce in the unit of mass the unit acceleration. If we keep to the units of length and time already adopted, the unit acceleration is a velocity of one foot in a second gained in a second. Now if we choose any force as our unit of force then the unit of mass is fixed by the equation of Art 75. It is that quantity of matter to which the unit force will communicate a velocity of one foot per second in a second. If on the other hand we choose a unit of mass, then the unit of force is fixed by the equation. It is such a force as will communicate to the quantity of matter chosen as the unit of mass, the unit acceleration.

80. Let the unit of force be first chosen, and since weight is a very usual measure of force let the weight say of one pound be selected as the unit of force, then the unit of mass is determined by the equation of Art 75. It is the quantity of matter to which a weight of one pound will communicate an acceleration of one foot per second. Now if the weight of one pound act on the quantity of matter in one pound, it will produce (Art. 76) an acceleration of g feet per second, and therefore to produce an acceleration of only one foot per second, we must take g times as large a mass, that is a mass of g pounds. Hence the unit of mass is the quantity of matter in g pounds, when the weight of one pound is the unit of force.

Similarly if one gramme be taken as the unit of force the unit of mass is the mass of g grammes.

Of course any weight whatever may be selected as the unit of force, and the corresponding unit of mass obtained by the equation. Such a unit force is called a *gravitation* unit. The British gravitation unit of force usually adopted is the weight of one pound, and the metrical gravitation unit, the weight of one gramme.

81. Gravitation units are very convenient in practice, and they are correct so long as we keep to the same place. But the force of gravity is different in different latitudes, and at different elevations above the sea level. The weight of one pound is therefore not a constant force, and in order to know its value at any place, we must know the acceleration of gravity at that place. Gravitation units are consequently not invariable, and are therefore inconvenient when we wish to compare forces at different places. For this purpose invariable units are required. We can obtain such units by first selecting the unit of mass. Since the mass of a body is constant, no matter where its situation, the unit of force which will be obtained by the equation of Art. 75 will also be invariable. The unit of force thus obtained is called an *absolute* or *kinetic* unit.

82. Let the unit of mass then be first chosen, and let it be the quantity of matter in one pound. The unit of force

must be such a force as will produce in this mass an acceleration of one foot per second. Now the weight of one pound will produce in this mass an acceleration of g feet per second, therefore the force that will produce an acceleration of 1 foot per second, must be the $\frac{1}{g}$ th part of the weight of one pound. The unit force is therefore a force equal to the weight of $\frac{1}{g}$ lb.

Similarly if the mass of one gramme be the unit of mass the unit of force is $\frac{1}{g}$ gramme.

These units of force are invariable, and are usually called *absolute* or *kinetic* units. The weight of 1 lb. varies with the place, but the value of g varies in the same ratio, and therefore the quotient $\frac{1}{g}$ is constant.

Any quantity of matter may be chosen for the unit of mass, and the absolute unit of force can then be found by the equation.

The British kinetic unit of force adopted is the weight of $\frac{1}{g}$ lb. It is now usually called the Poundal. The Metrical kinetic unit is the weight of $\frac{1}{g}$ gramme. It is called the Dyne. The poundal is the force which will give to the mass of one pound an acceleration of one foot per second. The dyne is the force which will give to the mass of one gramme an acceleration of one centimetre per second.

The British Association Committee on Units have recommended the general adoption of the Centimetre, Gramme, and Second as the fundamental units of length, mass, and time respectively; and that the absolute units derived from these be called the C.G.S. units.

83. The British standard of mass is a mass of platinum deposited in the office of the Exchequer and defined by Act

of Parliament to be the Imperial Standard Pound Avoirdupois. Its one seven-thousandth part is the grain. The French standard of mass is the "Kilogramme des Archives," also made of platinum. Its one-thousandth part is the gramme. The kilogramme contains 15432·34874 grains.

84. The value of g varies in different places. Its mean value for Great Britain, expressed in British units of length and time, may be taken as 32·2 feet per second, and in metrical units as 981·4 centimetres per second. The value of a poundal is therefore somewhat less than half an ounce. It is equal to 13825·38 dynes.

85. The chief statements of the previous articles may be exhibited in the following form:—

Units of Force.	Gravitation units.	Weight of one pound.
		Weight of one gramme.
	Absolute or Kinetic units.	The Poundal, or weight of $\frac{1}{g}$ lb.
		The Dyne, or weight of $\frac{1}{g}$ gramme.

From the above table it is seen that we can reduce quantities expressed in gravitation units to their equivalents expressed in kinetic units and conversely. To reduce pounds force to poundals, or to reduce grammes force to dynes at any place, we multiply by the value of g in that place; and to reduce poundals to pounds force or dynes to grammes force we divide by g.

86. *Momentum.*—We have seen that when the unit force acts on the unit mass it produces a unit velocity in the unit of time, and the body then possesses a quantity of motion which is taken as the unit quantity of motion. If m units of mass move with the unit velocity they will possess m units of quantity of motion; and if m units of mass move with v units of velocity they will possess mv units of quantity of motion. The term *momentum* is employed to

denote quantity of motion, and hence if m denote the mass of a body and v its velocity its

$$\text{momentum} = mv.$$

87. *Measure of Force.*—When a force acts on any mass for the unit time it produces a certain acceleration. If the mass be increased the acceleration is diminished in the same ratio, and if the mass be diminished the acceleration is proportionally increased; but the product of the mass and the acceleration, which expresses the quantity of motion or momentum generated in the unit time, is unaltered. This is that part of the Second Law of Motion which is expressed by the equation of Art. 75—

$$F = mf.$$

The quantity of motion produced by a force in the unit of time is invariable whatever be the mass of the body on which the force acts, and hence *the kinetic measure of a force is the momentum it can produce or destroy in a second.*

88. If the force acts for two seconds the momentum produced is evidently twice that in one second, and if for t seconds, t times that in one second. Thus since—

$$F = mf$$
$$\therefore Ft = mft$$
$$\therefore Ft = mv. \quad \text{Since, Art. 12, } ft = v.$$

Hence if the force act on the body for a time t, the whole effect of the force is measured by the momentum produced.

89. We may assume that whatever momentum is generated by a given force acting for a given time, the same momentum will be destroyed in the same time by the same force acting as a resisting force. Hence if the momentum be given and the resisting force, the *time* in which the momentum will be destroyed is given by the preceding equation—

$$t = \frac{mv}{F}, \text{ or}$$

$$\text{Time} = \frac{\text{Momentum}}{\text{Resistance}}.$$

90. The principles that have been explained will enable

us to find the acceleration when a given force acts on a given mass. We shall consider some cases and determine in each the general expression for the acceleration. These formulæ will be found of great service in the solution of problems.

(1) *To find the acceleration when a force acts on a body which is perfectly free to move in the direction in which the force acts.*

Let the force be P acting on a body whose weight is Q. We make use of Art. 78. The weight Q would generate in the mass of Q an acceleration g, therefore P will produce in the same mass an acceleration which is given by the proportion of Art. 78;

$$Q : P :: g : f \therefore f = \frac{P}{Q}g.$$

(2) *To find the acceleration when two bodies P and Q are connected by a string passing over a pulley, as in Atwood's Machine* (Art. 71).

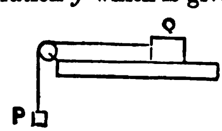
The force which produces motion in the system is $P - Q$, and the mass moved is the mass of $P + Q$. The weight $P + Q$ would produce an acceleration g in the mass of $P + Q$, if the bodies were allowed to fall freely, therefore the force $P - Q$ will produce in the same mass an acceleration f, which is given by the proportion of Art. 78;

$$P + Q : P - Q :: g : f \therefore f = \frac{P - Q}{P + Q}g.$$

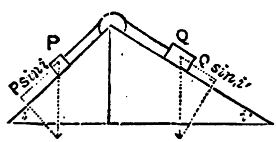
(3) *To find the acceleration when a weight P hanging vertically draws a weight Q along a smooth table by means of a string passing over a pulley at the edge of the table.*

The force which produces motion in the system is P, and the mass moved is the quantity of matter in $P + Q$. The weight of $P + Q$ would produce in this mass an acceleration g, therefore P will produce an acceleration f which is given by Art. 78;

$$P + Q : P :: g : f \therefore f = \frac{P}{P + Q}g.$$



(4) *To find the acceleration when a weight P placed on an inclined plane whose inclination is i draws a weight Q up another inclined plane whose inclination is i′, by means of a string passing over a pulley at the common vertex of the planes.*



Resolving P and Q along their respective planes, the resolved parts are P sin i and Q sin i'. The force producing motion in the system is therefore P sin i − Q sin i' and the mass moved is the mass of P + Q. Hence by Art. 78,

$$P+Q : P\sin i - Q\sin i' :: g : f \therefore f=\frac{P\sin i - Q\sin i'}{P+Q}g.$$

91. The general equations expressing the relations between space, time, and acceleration are given in Arts. 11 and 12, Chap. II. These will apply to any case of uniformly accelerated velocity. By substituting in these equations the value for f in any of the foregoing cases we can determine for that case the relations between the space described, the time of motion, and the acceleration due to the force acting on the system.

We proceed to consider some problems illustrating the foregoing principles.

Examples.

1. A force of 5 lbs. acts on a mass whose weight is 40 lbs.: what space will be described in 10 seconds?

$$\text{Art. } 90, f=\frac{P}{Q}g=\frac{5}{40}32=4$$

$$s=\tfrac{1}{2}ft^2=\tfrac{1}{2}\times 4\times 10^2=200 \text{ ft.}$$

2. Weights of 33 lbs. and 31 lbs. are connected as in Atwood's machine: after the system has been in motion for 5 seconds the string breaks: where will each body be at the end of the next 10 seconds?

$$\text{Art. } 90, f=\frac{33-31}{33+31}32=1 \text{ ft. per sec.} \quad v=ft=1\times 5=5 \text{ ft.}$$

Therefore one body has an initial downward velocity and the other an upward velocity of 5 ft. per sec. when the string breaks.

$$s = u\,t + \tfrac{1}{2}\,g\,t^2 = 5 \times 10 + \tfrac{1}{2} \times 32 \times 10 = 1650 \text{ ft.}$$

$\therefore$ The body moving downwards will be 1650 ft. below its position when the string broke.

$$s = u\,t - \tfrac{1}{2}\,g\,t^2 = 50 - 1600 = -1550 \text{ ft.}$$

$\therefore$ The body moving upwards will be 1550 feet below its position when the string broke.

3. A body weighing 50 lbs. is acted on by a constant force which acts for 5 seconds and then ceases to act: the body moves through 60 feet in the next 2 seconds. Express the force in absolute units.

$$\text{Velocity} = \frac{60}{2} = 30 \text{ ft.} \qquad \text{Acceleration} = \frac{v}{t} = \frac{30}{5} = 6 \text{ ft.}$$

$$f = \frac{P}{Q}.g \;\therefore\; 6 = \frac{P}{50}\,g \;\therefore\; P = \frac{50 \times 6}{g} \text{ in gravitation units}$$

and $P = 50 \times 6 = 300$ absolute units.

4. A body weighing 24 lbs. having a velocity of 100 feet per second is resisted by a force of 1 lb.: how long will it move?

$$\text{Art. 89, time} = \frac{\text{momentum}}{\text{resistance}} = \frac{m\,v}{R} = \frac{\frac{24}{g}.\,100}{1} = 75 \text{ secs.}$$

5. Two weights of 8 lbs. each are connected as in Atwood's machine: a bar weighing 2 lbs. is placed on one of the weights and motion takes place till that weight has descended 16 feet: find (*a*), the time taken; (*b*), the velocity acquired.

By Art. 90 the acceleration $f = \frac{10-8}{10+8}.\ 32 = \frac{32}{9}$

From Art. 12 equation 2, $t = \sqrt{\frac{2s}{f}} = \sqrt{\frac{2 \times 16}{\frac{32}{9}}} = \sqrt{9} = 3$ secs. (*a*.)

From Art. 12 equation 3, $v = \sqrt{2fs} = \sqrt{2 \times \frac{32}{9} \times 16} = \frac{32}{3} = 10\frac{2}{3}$ ft. (*b*.)

Exercises.

1. A force of 10 lbs. acts on a mass whose weight is 40 lbs.: what is the acceleration?

2. Weights of 17 ozs. and 15 ozs. are attached as in Atwood's machine: find the acceleration, and the space described by each body in 5 seconds.

3. In the foregoing case when each weight has moved from rest through a space of 10 feet, what velocity has it acquired?

4. In the foregoing, if the system has been in motion for 3 seconds, with what velocity is each body moving?

5. A weight of 10 lbs. hanging over the edge of a smooth table draws a weight of 30 lbs. along the table: what space will be described by each body in 2 seconds?

6 If in the foregoing case, when the system has been in motion for 2 seconds, the string breaks: find the space described by each body in the next 3 seconds.

7. A mass whose weight is 8 lbs. under the action of a single constant force moves from rest through a space of $2\frac{1}{2}$ feet in the first second: what is the magnitude of the force?

8. If the measure of the force of 16 lbs. weight be 16, what will be the measure of the mass of 16 lbs.?

9. A force of 5 lbs. acts on a mass of 5 lbs., a force of 5 lbs. on a mass of 20 lbs, and a force of 20 lbs. on a mass of 5 lbs.: what are the accelerations?

10. A force of 7 lbs. acts on a mass of 1 cwt.: find the velocity produced in 5 seconds, and the space described in 10 seconds.

11. A body whose mass is 8 lbs. is known to be under the action of a single constant force. It is observed to move from rest and in the first second of its motion to describe a distance of 5 feet: what is the magnitude of the constant force?

12. A body containing 50 lbs. of matter is set in motion by a constant force, which acts for 5 seconds in the direction of the motion: it then ceases to act, and the body (now acted upon by no external force) moves over 60 feet in the next two seconds. Find (1) the acceleration, (2) the magnitude of the force in gravitation units, (3) the magnitude of the force in kinetic units.

13. A body is thrown upwards with a velocity of 96 feet per second: after how many seconds will it be moving *downwards* with a velocity of 40 feet per second?

14. A weight of 100 lbs. is drawn along a smooth horizontal table by a weight of 20 lbs. hanging vertically: find the space described in 5 seconds.

15. Two equal weights each of 50 ozs. are connected by a string passing over a pulley: what weight must be added to one of them to make it descend 64 feet in 5 seconds?

16. A weight of 10 lbs. is suspended from one end of a string: find the weight which must be attached to the other end, so that when the string passes over a fixed pulley, the acceleration must be half that of gravity.

17. A weight of 10 lbs. draws another weight of 8 lbs. by a string passing over a pulley: what is the acceleration?

18. What force acting parallel to a horizontal plane will move a body weighing 18 lbs. along the plane with an acceleration of 5 feet per second; the friction being equal to 2 lbs.?

19. Two equal weights of 50 lbs. each are connected as in Atwood's machine: what weight must be taken from one and added to the other so that the heavier will descend through 20 feet in $2\frac{1}{2}$ seconds?

20. A weight of 8 lbs. hanging vertically draws a weight of 12 lbs. along a smooth horizontal table by means of a string passing over the edge of the table: find the space described by the body on the table in the third second of its motion.

21. Weights of 12 lbs. and 4 lbs. are placed on two inclined planes, whose inclinations are 60° and 30° respectively, and are connected by a

string which passes over a pulley at the common vertex of the planes: find the velocity acquired by each body in 4 seconds after the system has been started from rest.

22. In the foregoing case what space will be traversed from rest in 4 seconds by each body.

23. The speed of a railway train increases uniformly for the first 3 minutes after starting and during this time it travels 1 mile. What speed (in miles per hour) has it now gained, and what space did it describe in the first 2 minutes?

24. In the last question supposing the line level and disregarding friction and the resistance of the air, compare the force exerted by the engine with the weight of the train.

25. A weight of 6 ozs. is drawn up along the lid 4 feet long and rising 2 in 9 of a smooth desk by a weight of 5 ozs., which attached to the other weight by a string hangs over the top of the desk and descends vertically. Find the velocity acquired when the heavier weight reaches the top of the desk.

26. A horse exerting a pull of 300 lbs. draws a waggon whose weight is half a ton up a hill whose height is 20 feet and length 200 feet: find the time of ascending the hill, friction being neglected.

27. Through what distance must a force equal to the weight of 1 lb. act on a body weighing 16 lbs. so as to increase its velocity from 20 feet to 30 feet per second?

28. A stone weighing 12 lbs. is thrown along ice with a velocity of 64 feet per second, and comes to rest in 18 seconds: what is the force of friction?

29. Two weights of 3 lbs. and 4 lbs. are connected by a string passing over a pulley. A bar weighing 2 lbs. is placed on the 3 lbs. weight, and after motion has continued for 3 seconds the bar is removed: for what time longer will the 3 lbs. weight descend before coming to rest, friction being neglected?

30. In the foregoing case what will be the velocity of the system after 7 seconds from the instant it came to rest?

31. If in question 29 the motion had continued for 9 seconds before the bar was removed, how long afterwards would the 3 lbs. descend?

32. What is the momentum of a moving body? How is it estimated?

33. In a vacuum light bodies and heavy bodies fall with the same velocity: show from dynamical principles that their velocities must be equal.

34. If the mass of 1 lb. be the unit of mass, and 1 foot and 1 second be the units of space and time, how would you define the unit of force, and how many such units of force are there in the weight of $1\frac{1}{2}$ lbs.?

35. Find the unit of mass, when the unit of force is 100 lbs. weight, the unit of length 2 feet, and the unit of time a quarter of a second.

36. Find the unit of force when the unit of mass is the mass of 1 ton, the unit of length 1 yard, and the unit of time 1 minute, g being 32 foot-second units.

CHAPTER VII.

REACTIONS—THIRD LAW OF MOTION.

General Remarks on Laws of Motion.

92. Up to the present we have taken only a partial view of the action between two bodies; we may now consider that action more fully.

If a body be at rest upon a horizontal table, there is a mutual action between the body and the table which may be regarded from two different aspects, and, as Clerk Maxwell has pointed out, what we call *force* is one aspect of this mutual action. Newton employed the term *action* to denote the force exerted by one body upon the other, and *reaction* the force exerted by the latter upon the former. In recent works the term *stress* has been used to denote the whole of the mutual action between bodies.

93. The stress is called by different names according to the different modes in which the action takes place. It is a *pressure* if the bodies are in contact and the action on each is directed away from the other. It is a *tension* if the bodies are connected by a string, and the action is towards each body from the other. It is *attraction* or *repulsion* if the bodies act on each other at a distance without our being able to perceive any intermediate body by which the action is effected.

94. The forces of action and reaction being only different aspects of the same stress, must of course be always equal; and the statement of this fact forms Newton's THIRD LAW OF MOTION.

THIRD LAW. *To every action there is an equal and opposite reaction, that is the actions of two bodies upon each other are always equal and in opposite directions.*

95. The following illustrations of the law are given by Newton:—

If anyone presses a stone with his finger, his finger is equally pressed by the stone.

If a horse draw a stone by means of a rope, the horse is pulled back towards the stone with an equal force.

If one body attract another, the latter exerts an equal attraction on the former

When one body impinges on another, whatever change in the quantity of motion is produced in one, an equal and opposite change is produced in the momentum of the other. The velocities produced in this case will also be in opposite directions, but will not be equal unless the masses be equal. When the masses are unequal, the velocities due to the action will be inversely as the masses.

96. Examples of reactions where motion does not occur will be examined under Statics. As illustrating the Third Law of Motion we shall here consider some cases of Tensions, and of Pressures upon planes in motion.

The subject of Impact or the Collision of bodies will be treated of in the next chapter.

97. *To find the tension of the string in Atwood's machine.* —Let P and Q be the weights of the two bodies (See fig. of Art. 71), and let the mass of the pulley and the weight of the string be neglected. Let T be the tension, which is uniform throughout the string. Then if P is moving downwards and Q upwards, the resultant force acting on P is $P-T$, and the mass moved by this force is the mass of $P=\frac{P}{g}$ (Art. 76). Similarly the resultant force on Q is $T-Q$, and the mass moved is $\frac{Q}{g}$. Dividing the forces by the respective masses we obtain the accelerations (Art. 75), and as the bodies are connected together, the accelerations are equal; therefore

$$\frac{P-T}{\frac{P}{g}}=\frac{T-Q}{\frac{Q}{g}};\text{ and solving for T, we find } T=\frac{2PQ}{P+Q}.$$

98. *To find the tension of the string when a weight P hanging vertically draws another weight Q along a smooth table.* (See Art. 90.)—Let T = the tension. Then the resultant force acting on P is $P - T$, and the mass moved by this force is $\frac{P}{g}$. The resultant force on Q is T (since the weight of Q is equilibrated by the reaction of the table), and the mass is $\frac{Q}{g}$; therefore as in last Art.

$$\frac{P-T}{\frac{P}{g}} = \frac{T}{\frac{Q}{g}} \text{ and } T = \frac{PQ}{P+Q}$$

99. *To find the tension of the string in the conditions of Art.* 90 (4).—Let T = the tension. The resultant force acting on P is $P \sin i - T$, and the mass is $\frac{P}{g}$. The force on Q is $T - Q \sin i'$, and the mass is $\frac{Q}{g}$. Therefore

$$\frac{P \sin i - T}{\frac{P}{g}} = \frac{T - Q \sin i'}{\frac{Q}{g}}; \text{ therefore } T = \frac{P\,Q\,(\sin i + \sin i')}{P+Q}$$

100. *To find the tension of a string hanging vertically from a point of support, and having a weight attached to its lower end: (a.) When the point of support is at rest or is ascending or descending with uniform velocity: (b.) When ascending or descending with an accelerated velocity.* And—

To find the pressure of a weight resting upon a plane when the plane moves as stated in (a.) and (b.) respectively.

If a body be attached to the lower end of a string, the upper end of which is supported, and if the system be at rest, the string, by the Third Law, is stretched with a force equal to the weight of the body. If now a uniform velocity upwards or downwards be given to the point of support, the body when it has acquired the velocity will, by the First

Law, tend to move uniformly with that velocity, and, hence, the string will be stretched with the same force as when the point of support and the body are at rest.

The same results hold in the case of a body resting on a plane. If W be the weight of a body placed on a horizontal plane, the pressure on the plane and the reaction will each be W, whether the plane be at rest or be in motion with uniform velocity upwards or downwards. The weight W, if free to move, would produce in the mass of the body an acceleration g (Art. 76), and therefore the reaction of the plane is equal to a pressure W or an acceleration g upwards. If now the plane, or the point of support of the string, move upwards with an *accelerated* velocity, the pressure on the plane, or the tension of the string is increased by a pressure or tension which corresponds to the increase of acceleration upwards. If, for instance, the plane bearing the weight W move upwards with a velocity increasing by an acceleration $\frac{1}{4}g$, then since if the plane were at rest the reaction W upwards would be equivalent to an acceleration g, the acceleration $g + \frac{1}{4}g$ is equivalent to a reaction of $W + \frac{1}{4}W = \frac{5}{4}W$, which is therefore the pressure on the plane. If the plane move downwards with a velocity which increases by the acceleration $\frac{1}{8}g$, then the pressure W is diminished by the pressure corresponding to this acceleration. Hence the pressure is $W - \frac{1}{8}W = \frac{7}{8}W$. In this case the plane prevents an acceleration of only $\frac{7}{8}g$, and therefore the pressure is only $\frac{7}{8}W$. If, again, the plane descend with a velocity which increases by g, it is evident that it will experience no pressure from W, which will move downwards just in contact with but without pressing on the plane.

General Remarks on the Laws of Motion.

101. In the preceding chapters the Laws of Motion have been separately explained and illustrated. We shall here repeat these laws and make some general remarks upon them.

First Law.—*Every body continues in its state of rest or of uniform motion in a straight line unless compelled by impressed forces to change that state.*

Second Law.—*The change in the quantity of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

Third Law.—*To every action there is an equal and opposite reaction, that is, the actions of two bodies upon each other are always equal and in opposite directions.*

102. Although, as has been already stated, these laws are deductions from observation and experiment, yet they do not admit of direct experimental proof. They may be regarded as fundamental hypotheses, and their truth is established by finding that in all cases the effects observed agree with the theoretical deductions from these principles. Thus in astronomy the observed motions and positions of the heavenly bodies are invariably found to agree with the results obtained by calculation.

103. The Laws of Motion might more strictly be called the Laws of Force and Motion. The First Law tells us when a force is in action; the Second enables us to measure it; the Third points out its nature.

104. *First Law.*—In the First Law the motion and the rest refer to the body as a whole, and not to its particles. The law is the statement of that property of matter which has been named Inertia. A body will remain in its state of rest or of uniform motion in a right line unless some force act on it; and, hence, if a change in its state whether of rest or motion is observed, we infer that a force acts on the body. The law thus furnishes us with a definition of *force.* Force *is any cause which alters or tends to alter a body's state of rest or of uniform motion in a straight line.*

Again, since a body in motion and not acted upon by forces will continue in uniform motion, and will, therefore, pass over equal spaces in equal times, the law also gives us a definition of *equal times.* Equal times are those in which a body in motion, and not acted upon by force, passes over equal spaces.

105. *Second Law.*—The Second Law is the statement of the relations between Force, Mass, and Acceleration. These relations have been explained at length in Chap. VI. From Arts. 86 and 87 it will be seen that the statements of Art. 70 are merely a paraphrase of the Second Law.

The law furnishes definitions of *equal forces*, and of *equal masses.*

Equal forces are those which communicate to the same mass equal accelerations.

Equal masses are those to which the same force communicates equal accelerations.

What the law does *not* state requires as much attention as what it does state, and from its silence very important inferences may be drawn.

No mention is made of the state in which the body is in, and the law is true whether the body be at rest or in motion. There is nothing stated about other forces acting upon the body, and the law applies, whether other forces act on the body or not.

Hence, when any forces act on a body each force produces an acceleration the same in magnitude and direction as though no other forces acted on the body. This principle was assumed in Arts. 42 and 56.

If any number of forces act on a body each communicates an acceleration proportional to itself in magnitude, and in the same direction. Hence, if lines be drawn representing the accelerations in magnitude and direction, these lines will also represent the forces, and if the resultant acceleration be determined by Art. 33, the line so found will also represent the resultant force. Forces may therefore be compounded in the same way as accelerations. In Statics we shall make use of this method for obtaining the resultant of a number of forces.

106. *Third Law.*—The Third Law is the statement of the mutual action between bodies. If two bodies act on each other the change in the momentum of one is equal and opposite to the change in the momentum of the other. The respective velocities are opposite in direction, and inversely

proportional to the masses. The law is not confined to the action between two bodies in contact. It is true of the attraction of gravity. In the case of a stone falling to the ground, the force exerted by the stone on the earth is exactly the same as that of the earth upon the stone. And the law equally applies to all cases of magnetic and electric attractions and repulsions.

In the chapter on Energy it will be shown that another meaning may be given to the terms action and reaction, and that thus the Third Law may have a still wider and more important application.

EXAMPLES.

1. Two weights of 20 lbs. and 25 lbs. are connected by a string passing over a pulley: find the tension of the string when the system is in motion.

$$\text{Art. 97,}\quad T=\frac{2PQ}{P+Q}=\frac{2\times 20\times 25}{20+25}=22\tfrac{2}{9}\text{ lbs.}$$

2. A weight of 112 lbs. rests on a lift: find the pressure on the lift, (*a*) when it is ascending or descending uniformly; (*b*) when it is ascending with a velocity which increases by 4 feet per second; (*c*) when it is descending with a velocity which increases by 8 feet per second; (*d*) when it is descending with a velocity which diminishes by 8 feet per second.

Art. 100. (*a*) 112 lbs.
(*b*) 32 : 36 :: 112 : 126 lbs.
(*c*) 32 : 24 :: 112 : 84 lbs.
(*d*) 32 : 40 :: 112 : 140 lbs.

3. A balloon ascends with a uniformly accelerated velocity so that a weight of 1 lb. produces on the hand of the aeronaut sustaining it a downward pressure equal to that which 17 ozs. would produce at the earth's surface: find the height which the balloon will have attained in one minute from the time of starting, not taking into account the variation of the accelerating effect of the earth's attraction.

The upward acceleration produces an increase of $\frac{1}{16}$th in the pressure of the weight, therefore (Art. 100) the acceleration $=\frac{1}{16}$th of $32=2$.

The space described in one minute with an acceleration 2, is obtained from the equation $s=\frac{1}{2}ft^2$

$$s=\tfrac{1}{2}\times 2\times 60^2=3600\text{ feet.}$$

Exercises.

1. One of the bodies in Atwood's machine is 10 lbs., and the string is able to bear a strain of 12 lbs.: what is the weight of the other body so that the string may be just on the point of breaking?

2. A weight of 25 lbs. hanging vertically draws a weight of 100 lbs. along a smooth horizontal table: what is the tension in the string?

3. If the weight on a smooth horizontal plane be 50 lbs., what weight hanging vertically will produce a tension in the string of 5 lbs. when the system is in motion?

4. A cannon ball weighing 100 lbs. leaves the gun, which weighs 1 ton, with a velocity of 1,000 feet per second: what is the velocity of the recoil of the cannon?

5. A shell weighing 40 lbs. and moving with a velocity of 100 feet per second bursts into two equal fragments. One of them moves in the same direction with the velocity of 160 feet per second: what is the velocity of the other?

6. The engine at the mouth of the shaft of a mine winds up the cage, which weighs $\frac{1}{2}$ ton, (1) with a uniform velocity of 10 feet per second, (2) with a velocity increasing by 10 feet per second: find in each case the tension of the rope attached to the cage.

7. Two weights of 50 grammes and 60 grammes are connected by a string passing over a pulley: find the tension of the string in gravitation units and in absolute units, respectively.

8. Find the tension of a rope which draws a carriage of 8 tons weight up a smooth incline of 1 in 5 and causes an increase of velocity of 3 feet per second.

9. The weights at the extremities of a string which passes over the pulley of an Atwood's machine are 500 grammes and 502 grammes. The larger weight is allowed to descend, and 3 seconds after motion has begun 3 grammes are removed from the descending weight. What time will elapse before the weights are again at rest?

10. A balloon carries a weight of 100 lbs. suspended by a cord from the car: find the tension of the cord, (*a*) when ascending uniformly at the rate of 16 feet per second; (*b*) when ascending with a velocity increasing by 16 feet per second; (*c*) when descending with a velocity increasing by 16 feet per second; (*d*) when descending with a velocity increasing by 24 feet per second; (*e*) when descending with a velocity increasing by 32 feet per second.

11. A man whose weight is 140 lbs. places himself on a lift: find his pressure on the lift, (*a*) when it is stationary; (*b*) when it is ascending or descending uniformly; (*c*) when it is ascending with a velocity which increases by the acceleration $\frac{1}{16}g$; (*d*) when ascending with a velocity which diminishes by the acceleration $\frac{1}{16}g$; (*e*) when descending with a velocity which increases at the rate of 32 feet per second; (*f*) when descending with a velocity which diminishes at the rate of 8 feet per second; (*g*) when descending with a velocity which diminishes at the rate of 32 feet per second.

12. A weight of 100 lbs. is attached to the lower extremity of a cord which hangs from the car of a balloon: find the tension in the cord, (*a*) when the balloon rises with a velocity increasing by 4 feet per second; (*b*) when it rises with a velocity decreasing by 4 feet per second.

13. I suddenly jump off a platform with a 20 lb. weight in my hand. What will be the pressure of the weight upon my arm while I am in the air?

14. A stationary engine draws by means of a rope a waggon weighing 10 tons up an incline of 30°: neglecting friction, find (*a*) the tension of the rope when the velocity is uniform; (*b*) when the waggon is moving up with an acceleration of 4 feet per second per second.

15. Weights of 12 ozs. and 9 ozs. rest on two inclined planes having a common height and inclinations whose sines are $\frac{1}{2}$ and $\frac{1}{3}$ respectively, the weights being connected by a string passing over the common vertex of the planes. Find the tension of the string when the system is in motion.

16. Two inclined planes have a common vertical edge, and the same inclination of 30° to the horizon. Weights of 20 ozs. and 15 ozs. rest on the planes, being connected by a string passing over a pulley at the common vertical edge: find the tension of the string when the system is in motion.

17. Show that from the expression for the tension in Art. 99, that in Art. 97 may be derived by supposing the planes to become vertical; and that the expression in Art. 98 may be obtained by supposing one of the planes to become horizontal and the other vertical.

18. A weight of 1 oz. hanging over the edge of a smooth horizontal table draws by means of a string a weight of 1 lb. along the table: find the tension of the string.

19. A body whose weight is 36 ozs. is drawn up an inclined plane whose inclination is 30° by an equal weight which hangs vertically, and is connected with the former by a string passing over a pulley at the vertex of the plane: find the tension of the string.

20. A body P is drawn up a plane whose inclination is i by a body Q which hangs vertically, the two bodies being connected by a string which passes over a pulley at the vertex of the inclined plane: find the tension of the string.

21. Show how the expression obtained in the foregoing exercise may be derived from that of Art 99.

[*] CHAPTER VIII.

IMPACT OR COLLISION OF BODIES.

107. Impact or impulse is the term employed to denote the action which lasts for a very brief period when two bodies come into collision. When, for instance, a cricket ball is struck by the bat, impulsive forces act upon both ball and bat in opposite directions for a very short time, and these by the Third Law are equal to each other. These forces do not differ in kind from those we have been considering. They differ only in the time of their action. We cannot, however, measure these forces by the method of Art 87, that is by the momentum generated in a unit of time, since the forces act only for an indefinitely brief period, the length of which is unknown. A different measure must, therefore, be adopted for impulsive forces, and it has been agreed to measure an impulsive force by the whole momentum it produces. This measure will enable us merely to compare impulsive forces with each other, but not to compare them with ordinary forces. This, however, will not lead to any difficulty, because in any case where ordinary and impulsive forces act together the effects of the former would be so small during the extremely brief period of the action of the latter that they may be neglected, and the impulsive forces alone will remain to be considered.

108. The nature of these forces will be illustrated by considering a particular case of impact. Let us suppose two ivory balls in motion, either in the same or in opposite directions, to come into direct collision. For a very brief period a portion of each ball suffers compression and becomes flattened, and it may be assumed that this action continues until the velocities of the balls become equal, while during

this period certain impulsive forces act in opposite directions upon each ball. Then, owing to the property of bodies called elasticity, the ivory tends to regain its original form, and whilst restitution is taking place it may further be assumed that another impulsive force acts on each body in the same direction as before. The impulsive force in each case is measured by the momentum lost by the striking ball and gained by the ball struck. If the bodies were perfectly elastic the force during restitution would be equal to that during compression. If the bodies were perfectly inelastic there would be no restitution, and no second impulsive force, and the bodies would remain together after the first period of impact. As all bodies are imperfectly elastic, the second impulsive force is always less than the first.

109. A similar explanation will apply to the case of a ball impinging on a fixed plane. The velocity of the ball becomes 0 at the end of the first period of impact, and the momentum lost by the ball is given to the plane, which being fixed is a portion of the earth, and is therefore given up to the earth. The ratio of the impulsive force during the second period of impact to that during the first period is different for different bodies, but is the same for the same two bodies.

110. The impulsive forces which act during compression and restitution cannot be determined directly by experiment, but their ratio can be inferred from the observed velocities before and after impact. When for instance a ball impinges on a fixed plane the ratio of the velocity of approach to that of recoil is the same as the ratio of the momentum in approach to that in recoil, and is therefore the same as the ratio of the momentum lost during compression to that gained during restitution, or finally as the ratio of the impulsive force during compression to that during restitution. When both bodies are in motion, the velocity of approach or of recoil is their *relative velocity*, that is the algebraic difference of the velocities of the bodies. Newton was the first to show by experiment that the ratio of the velocity of approach to that of recoil is constant for the same two

bodies, but is different for different bodies. This ratio for any two substances is called the *coefficient of restitution*, or the *coefficient of elasticity*, or the *modulus of elasticity*. It is usually denoted by e. More recent experiments have shown that the coefficient of elasticity for any two bodies is constant only for small velocities of approach.

111. Hence if after impact bodies recoiled with a velocity equal to that of approach, they would be *perfectly elastic*; if with a less velocity they would be *imperfectly elastic*; and if they did not recoil they would be *perfectly inelastic*. The coefficient of elasticity for perfectly elastic bodies is 1, for imperfectly elastic bodies is a fraction less than 1, and for perfectly inelastic bodies is 0. No bodies in nature are found perfectly elastic or perfectly inelastic. Bodies such as glass, marble, ivory are very elastic, and those such as putty or soft clay are very inelastic.

112. If a body with a velocity u impinge on a fixed plane, and if the coefficient of elasticity for the two substances be e, then denoting the velocity of recoil by v,

$$v = -eu,$$

where the $-$ sign indicates that the velocities are in opposite directions.

113. If both bodies be in motion with velocities u and u' respectively, and if v and v' denote the respective velocities after impact, and e denotes the coefficient of elasticity. then the difference of the velocities before the time of impact will be the velocity of approach, and the difference of the velocities after impact will be the velocity of recoil. Hence

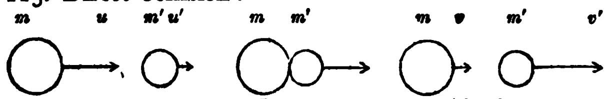
$$v - v' = -e(u - u').$$

114. We now proceed to consider some cases of the collision of bodies. For the sake of simplicity we shall suppose that these bodies are spheres.

Impact of Bodies in Motion.—There are two cases of the collision of spheres in motion: (1) Direct Collision; (2) Oblique Collision. The collision is called direct when the centres of the spheres are moving in the straight line in

which the impact takes place. It is called oblique when the centres are moving in different lines.

115. Direct Collision :—



Before impact. Impact. After impact.

Let the spheres whose masses are m and m' be moving with their centres in the same line with velocities u and u' respectively. Let m overtake and impinge on m', and after impact let the velocities be v and v' respectively. Then the whole momentum lost by the striking sphere is $m\ (u - v)$, and the whole momentum gained by the struck sphere is $m'\ (v' - u')$. These are equal by Newton's Third Law, therefore

$$m\ (u - v) = m'\ (v' - u').$$

Again $u - u'$ is the velocity of approach, and $v - v'$ is the velocity of recoil, and (Art. 113)

$$v - v' = -e\,(u - u').$$

Solving these equations for v and v' we find

$$v = \frac{mu + m'u' - em'\ (u - u')}{m + m'},$$

$$v' = \frac{mu + m'u' + em\ (u - u')}{m + m'}.$$

If we suppose the bodies perfectly elastic and equal, then $e = 1$, and $m = m'$, and we have from the foregoing equations

$$v = u' \text{ and } v' = u.$$

Hence such bodies would interchange velocities by the collision.

If the bodies move in opposite directions before impact, we have only to change the sign of u' in the above equations n order to get the velocities after impact.

If we suppose the bodies perfectly inelastic, then $e = 0$, and we have from the equations

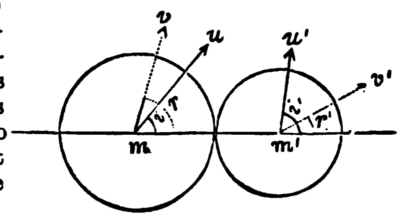
$$v = v' = \frac{mu + m'u'}{m + m'}.$$

116. This result may be obtained very readily from first principles as follows. Since the momentum lost by one body is gained by the other, it follows from the Third Law that the momenta before and after impact are equal. Again, since the bodies are inelastic, there will be no force tending to separate them after collision, and they will therefore move with a common velocity after impact. Let v be this common velocity. Then the momentum of the system after impact is $mv + m'v$, and before impact is $mu + m'u'$, and these are equal; thus

$$mv + m'v = mu + m'u'$$

$$\therefore v = \frac{mu + m'u'}{m + m'}.$$

117. *Oblique Collision.*—Let two spheres whose masses are m and m' be moving with velocities u and u' respectively, the directions of these velocities being inclined to the line of impact passing through the centres of the spheres at angles i and i' respectively. After impact let the velocities be v and v' whose directions (indicated by the dotted lines) are inclined to the line of impact at angles r and r' respectively.



Resolve the velocities before and after impact in the direction of the line of impact and at right angles respectively. The resolved velocities at right angles to the line of impact are

$$u \sin i, \ v \sin r, \ u' \sin i', \ v' \sin r';$$

and in the line of impact the resolved velocities are

$$u \cos i, \ v \cos r, \ u' \cos i', \ v' \cos r'.$$

No impulsive force acts in the direction at right angles to the line of impact, therefore (Arts. 56 and 105) the velocities

in this direction are not altered by the collision, and hence the velocities in the direction at right angles to the line of impact are the same before and after impact.

$$\text{Therefore } v \sin r = u \sin i \qquad (1)$$
$$v' \sin r' = u' \sin i' \qquad (2)$$

Again, the velocities in the line of impact are unaffected by the components at right angles to the line. To the velocities in the line of impact we can apply the rules of direct collision. Proceeding exactly as in Art. 115, we obtain

$$v \cos r = \frac{mu \cos i + m'u' \cos i' - em'(u \cos i - u' \cos i')}{m + m'} \qquad (3)$$

$$v' \cos r' = \frac{mu \cos i + m'u' \cos i' + em(u \cos i - u' \cos i')}{m + m'} \qquad (4)$$

It will be seen that these results may be obtained from the equations of Art. 115 by writing $v \cos r$ for v, $v' \cos r'$ for v', $u \cos i$ for u, and $u' \cos i'$ for u'.

Dividing equations (1) by (3) and (2) by (4) we obtain the values of $\tan r$ and $\tan r'$, and thus the directions of the motions of the bodies after collision are known.

By squaring equations (1) and (3) and adding, and by squaring (2) and (4) and adding, we obtain the values of v^2 and v'^2 (since $\sin^2 r + \cos^2 r = 1$) and thus the velocities after collision are known.

118. Impact of a body upon a Fixed Plane.

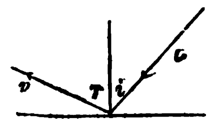
Direct Collision. If a body with a velocity u impinge perpendicularly on a fixed plane, then (Art. 112) it will recoil in the same line; and if the coefficient of elasticity be e, the velocity of recoil will be eu. Calling the velocity of recoil v, then since it is opposite in direction to that of approach we have

$$v = -eu.$$

If $e = 1$, $v = -u$. If $e = 0$, $v = 0$. That is if the bodies are perfectly elastic, the moving body recoils along the same line as it approached with a velocity equal to that of approach;

if the bodies are inelastic there is no recoil; and if imperfectly elastic the velocity of recoil is e times that of approach, and in the opposite direction.

119. *Oblique Collision.*



Let a body with a velocity u impinge on a fixed plane, and let the direction of the velocity before impact make an angle i with the perpendicular to the plane, and the direction of the velocity after impact make an angle r with the same perpendicular, and let the coefficient of elasticity be e.

Resolve the velocities along the plane and at right angles to it. The components along the plane are—

$$u \sin i \text{ and } v \sin r;$$

and at right angles to the plane, or in the line of impact they are—

$$u \cos i \text{ and } v \cos r.$$

No impulsive force acts on the bodies in the direction along the plane, and therefore the velocity before and after impact in this direction is unaltered.

Therefore—

$$v \sin r = u \sin i. \qquad (1)$$

To the components in the direction of the line of impact we can apply the laws of direct collision, therefore (Art. 112)—

$$v \cos r = -eu \cos i. \qquad (2)$$

Divide equation (1) by (2), and we find—

$$\tan i = -e \tan r. \qquad (3)$$

This gives the value of the angle r, and consequently the direction of the motion after impact is known.

Squaring equations (1) and (2) and adding, we obtain—

$$v^2 = u^2 (\sin^2 i + e^2 \cos^2 i). \qquad (4)$$

Thus the velocity after impact is known.

The angle i is called the angle of incidence, and r the angle of reflection.

The $-$ sign in equation (3) indicates that these angles are on opposite sides of the perpendicular.

If in equation (3) $e = 1$, then $\tan i = -\tan r$, and $v = u$. That is, if the elasticity be perfect, the angle of reflection is equal to the angle of incidence and the velocity after impact equal to that before.

If $e = 0$, then $r = 90^\circ$, and $v = u \sin i$. That is when the bodies are inelastic, the body impinging on the plane with velocity u will not recoil, but will run along the plane with the velocity $u \sin i$.

EXAMPLES.

1. Five perfectly inelastic bodies of equal masses are arranged at short distances along a straight line on a smooth horizontal plane. The first body is now caused to move in the same line with a velocity of 100 feet per second; it impinges on the second, then the two bodies impinge on the third and so on. Find the final velocity after the fourth impact.

Let the mass of each be 1, then, by Art 116,

$$\text{Velocity after 1st impact} = \frac{1 \times 100}{1+1} = \frac{100}{2}.$$

$$\text{,, \quad 2nd \quad ,,} = \frac{2 \times \frac{100}{2}}{2+1} = \frac{100}{3}.$$

$$\text{,, \quad 3rd \quad ,,} = \frac{3 \times \frac{100}{3}}{3+1} = \frac{100}{4}.$$

$$\text{,, \quad 4th \quad ,,} = \frac{4 \times \frac{100}{4}}{4+1} = \frac{100}{5} = 20 \text{ feet per sec.}$$

2. A body weighing 12 lbs. impinges directly on another at rest weighing 4 lbs., and by the impact loses $\frac{1}{3}$ of its velocity: what is the coefficient of elasticity?

From Art. 115, since $v = \frac{2}{3} u$ by the question, and $u' = 0$, and since the masses are proportional to the weights, therefore,

$$\tfrac{2}{3} u = \frac{12u - 4ue}{12+4} \therefore e = \tfrac{1}{3}.$$

3. A body falls from a height h on a horizontal plane; to what height will it rebound, the coefficient of elasticity being e?

Velocity acquired in falling through height $h = \sqrt{2gh}$.
Velocity of recoil (Art. 112), $= e\sqrt{2gh}$.

Greatest height attained with velocity $e\sqrt{2gh}$, (Art. 14),

$$= \frac{(e\sqrt{2gh})^2}{2g} = e^2h.$$

EXERCISES.

1. Two perfectly inelastic bodies of equal masses are moving in opposite directions with velocities of 40 feet and 32 feet per second, respectively: find their common velocity after impact.

2. A perfectly inelastic body moving with a velocity of 100 feet per second strikes another inelastic body of equal mass at rest: find the velocity after impact.

3. If in the preceding question the mass of the body at rest was four times that of the moving body: find the velocity after impact.

4. Equal perfectly inelastic balls are placed at short equal intervals in a smooth horizontal groove. The first is projected from one end along the groove with a velocity of 20 feet per second: find the velocities after successive impacts.

5. A body weighing 20 lbs. and moving with a velocity of 12 feet per second impinges on a body moving in the same direction, weighing 60 lbs. and having a velocity of 4 feet per second: find the velocities after impact, the coefficient of elasticity being $\frac{1}{2}$.

6. Two bodies of 15 lbs. and 60 lbs. are moving in opposite directions each with a velocity of 20 feet per second. If the coefficient of elasticity be $\frac{1}{3}$, find the velocities after impact.

7. A body weighing 12 lbs. and moving with a velocity of 4 feet per second meets a body of 3 lbs. moving in the opposite direction with a velocity of 16 feet per second: find the velocities after impact, (a) supposing the bodies perfectly inelastic; (b) supposing them perfectly elastic.

8. A body in motion strikes another at rest and by the impact loses one-fourth of its velocity. If the mass of the moving body be four times that of the body at rest, what is the coefficient of elasticity?

9. A perfectly elastic body moving with a velocity of 25 feet per second overtakes and strikes another perfectly elastic body of equal mass moving with a velocity of 15 feet per second: what are the velocities after impact?

10. A ball is allowed to fall on a horizontal plane from a certain height and rebounds to the height of 16 feet, the coefficient of elasticity being $\frac{1}{2}$. Another ball is allowed to fall on another horizontal plane from the same height and rebounds to the height of 9 feet. What is the coefficient of elasticity in the latter case?

11. A ball moving with a velocity of 10 feet per second strikes an equal ball at rest: if the coefficient of elasticity be $\frac{1}{2}$, what is the velocity of each ball after impact?

12. Four perfectly elastic balls of equal masses are placed at short distances in the same line. To the first ball a velocity of 20 feet per second is communicated in the line in which the balls are arranged: with what velocity will the last ball move off?

13. A perfectly elastic ball strikes another of equal mass at rest. At the instant of impact the direction of motion of the first ball makes an angle of 45° with the line joining the centres of the balls: what angle will the direction of motion of the same ball make with the same line after impact?

14. In the foregoing question find the angle which the direction of motion of the second ball after impact makes with the line joining the centres.

15. A ball falls from a height of 48 feet on a horizontal plane. If the coefficient of elasticity be $\frac{1}{4}$ to what height will the ball rebound?

16. A ball falls from a height of 96 feet upon a horizontal plane, rebounds, falls again and rebounds and so on: to what height will it reach on the third rebound, and to what on the fourth, the coefficient of elasticity being $\frac{1}{2}$?

17. A perfectly elastic ball moving with a velocity of 50 feet per second strikes another perfectly elastic ball of equal mass at rest. The angle between the directions of the motion of the striking ball before and after impact is 60°: what is the velocity of this ball after impact?

18. At what angle must a body strike a fixed plane so that the directions of its motion before and after impact may form a right angle, the coefficient of elasticity being $\frac{1}{3}$?

19. A body is projected from the floor of a room 12 feet high, strikes the ceiling, rebounds to the floor, and just reaches the ceiling again. If the coefficient of elasticity be $\frac{1}{3}$, find the velocity of projection.

20. Two perfectly elastic balls, one of which has three times the mass of the other, meet directly with equal velocities: find the velocity of the larger body after impact.

CHAPTER IX.

WORK AND ENERGY.

120. It has been shown that when a force acts upon a body for a given *time*, the effect is measured by the momentum produced; and this is expressed by the product of the mass which is moved into the velocity acquired in the given time.

Thus if a force F act on a mass m and generate a velocity v in a time t, then (Art. 88) $Ft = mv =$ momentum.

121. But it is also often necessary to consider a force acting through a given *space*, and the term *Work* is used to denote the effect produced when a force moves its point of application through any distance.

Work is done by a force when it moves its point of application in the direction in which it acts.

Work is done against a force when its point of application is moved in a direction opposite to that in which the force acts.

A force does work when it produces motion in a body and continues to act upon the body, whether that motion be opposed by resisting forces or not. Again, when a body in motion continues in motion through any space against a resisting force, work is done against the force.

122. Hence, for the production of the work there must be both force and motion. A weight resting on a table does no work, because there is no motion. A body not acted upon by forces and moving uniformly does no work, because there is no force. When a stone falls through any distance the earth does work in pulling the stone to the ground. A man does work when he lifts the stone through any height. A magnet does work when it draws a piece of iron towards it. Again, a bullet fired vertically upwards does

work against the force of gravity. A cannon ball striking and penetrating a clay bank does work in overcoming the resistance of the clay.

123. The term *Energy* is used to denote the power of doing work which is possessed by any body or system.

Energy is distinguished into two kinds, *Potential* and *Kinetic*.

A body or system though not actually doing work may possess the capability of doing it, and is consequently said to have potential or possible energy. A body which is in motion is said to have kinetic or moving energy. A stone resting on the top of a wall has potential energy from its position. If allowed to fall, it acquires motion, and possesses kinetic energy. During the time it is falling it has both potential and kinetic energy, the former diminishing and the latter increasing until it reaches the ground, when all its energy has been changed into kinetic. If the stone be thrown up with the velocity with which it struck the ground, it will reach the height from which it fell, and during its ascent it possesses both potential and kinetic energy, the latter diminishing and the former increasing till the body comes to rest at its greatest height, when its energy is once more entirely potential, and equal to the energy with which it started. Further, at any intermediate point the sum of the potential and kinetic energies possessed by the body is constant.

Thus one kind of energy can be changed into another, as one form of matter can be changed into another; but energy like matter is indestructible.

124. Since a force does work when it moves its point of application through any space, the amount of work done will vary with the magnitude of the force, and with the space moved through in the direction in which the force acts. Hence, if F denote the force and s the space, then

$$Fs = \text{work done.}$$

Since (Art. 40) force is measured statically by weight, a convenient estimate of the work done by a force is a weight

raised through a vertical height. Hence, if W be the weight of a body, and h the vertical height through which it is raised, Wh expresses the work done, and therefore

$$Fs = Wh.$$

This is the statical measure of the work accumulated in the body by the action of the force through the given space.

125. A kinetical measure of the work accumulated in a body may be obtained as follows :—

Let a force F act constantly through a space s on a body of mass m, and produce a velocity v. Let f be the acceleration, then (Art. 75) $F = mf$. But (Art. 12) when a body moves with a constant acceleration through a space s, and acquires a velocity v, $s = \frac{v^2}{2f}$. And since $Fs =$ the work done, therefore $Fs = mf.\frac{v^2}{2f}$, or $Fs = \frac{mv^2}{2}$.

Therefore $\frac{mv^2}{2}$ is the kinetical measure of the work accumulated in the moving body, and is usually called the *kinetic energy* of the body.

126. Thus if a force F act through a space s on a body of mass m, which is not acted upon by any resisting force, it will produce a moving energy expressed by $\frac{mv^2}{2}$. Conversely if a body of mass m be moving with a velocity v its kinetic energy is $\frac{mv^2}{2}$, and it will be capable of overcoming a force F through a space s. Thus every moving body can do work, and its energy is measured by the product of half its mass by the square of its velocity.

Similarly if a body whose weight is W fall through a height h, the work done is Wh. Now, $W = mg$, and

$h = \frac{v^2}{2g}$ (Art. 14) $\therefore Wh = \frac{mv^2}{2}$. Hence, the kinetic energy of the weight W after falling through a height h is $\frac{mv^2}{2}$, and if it now be made to do work, it will raise an equal weight to the same height from which it fell.

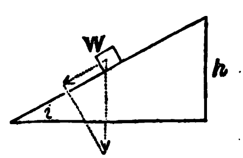
127. It may be assumed that the kinetic energy produced by a force acting through any space will be balanced by an equal resisting force acting through the same space. Hence, the space may be found through which a given moving energy will act against a resisting force.

From Art. 125,

$$Fs = \frac{mv^2}{2} \therefore s = \frac{\frac{1}{2}mv^2}{F}, \text{ or space} = \frac{\text{kinetic energy}}{\text{resistance}}.$$

128. In the foregoing the force is supposed to be constant, and the space described to be in the direction in which the force acts. If the direction in which the point of application moves be not in the line in which the force acts, then in order to find the work done, we resolve the force into two components, one in the line of direction and the other at right angles to it. The former is the working component, and the product of this component into the space described is the work done.

Similarly to find the work done *against* a resisting force when the motion is not in the line of action of the resistance we resolve the latter into two components, one in the direction of the resisting force and the other at right angles. The product of the former component into the length of path described is the work done against the resistance. Thus if a body whose weight is W be drawn up an inclined plane whose length is l and angle of inclination i, then the work done against gravity is $W \sin i \times l$.



If h = height of plane, $h = l \sin i$ and $Wh = Wl \sin i$. So that the work done is measured either by the product of

the whole weight into the vertical height through which it is raised, or by the product of the component in the line of action into the length of path described.

129. And since the component vanishes when a force is resolved in a direction at right angles to its line of action, it follows that there is no work done by or against a force when the motion is at right angles to the direction of the force. Hence, no work is done against gravity when a body is moved on a horizontal plane.

130. We have now established the two equations—

$$Ft = mv.$$
$$Fs = \tfrac{1}{2}mv^2.$$

The first gives an expression for the effect of a force acting for a given time, and the second an expression for the effect of a force acting through a given space. The first is the momentum, the second the energy. The momentum is proportional to the velocity, the energy to the square of the velocity. The momentum measures the effect of the force acting for a given time and may be called *accumulated force.* The energy is not a measure of the force, but of the work done by the force, and it may be called *accumulated work.*

131. Towards the close of the 17th century a great controversy arose among scientific men as to the proper measure of a force. Liebnitz and others adopted the erroneous theory that a force should be measured by the effect produced by its action through a given space; and they gave the name *vis viva* to the expression mv^2, which denotes the product of the mass of the body into the square of the velocity acquired in moving through the given space. The *vis viva* therefore is twice the kinetic energy. The term is now falling into disuse.

132. Conservation of Energy and Principle of Work.

When a force does work on a body it may produce an increase of either kinetic energy or potential energy or both. If no resisting forces act on the body, the work done is wholly applied to confer kinetic energy. Thus when a

stone falls through a vacuum the work done by gravity appears as kinetic energy in the stone. If, on the other hand, the stone be pulled upwards, suppose with a pulley, by a constant force equal to its weight, then whatever velocity upwards be communicated to the stone it will retain it, and will ascend with that uniform velocity; and the work done by the force is entirely expended in conferring potential energy upon the stone. If, again, the force with which the stone is pulled be greater than its weight, the stone will move upwards with an accelerated velocity, and it will gain both kinetic and potential energy. In all cases of work there is a transference of energy from one body or system to another, the amount of the energy given out by one being equal to that received by the other.

In the case where a body is moved against friction upon a horizontal surface with a force just sufficient to overcome the friction, the body does not acquire any increase of velocity, and it does not gain potential energy, and therefore it might be supposed that the work is expended without any resulting energy. Or, again, when a body falls through any height upon the ground its motion is destroyed when it strikes the ground, and apparently its kinetic energy is annihilated without any corresponding energy being produced. In such cases modern science has shown that the energy is not destroyed but that it changes its form and appears as *molecular* energy. The energy of the mass as a whole is transferred to the particles of the bodies which are set vibrating more rapidly, and this vibration of the particles is what we call *heat*. Heat is the energy possessed by the particles of a body. From numerous experiments it has been inferred that although the energy of one body or system may be transferred to another, and the mechanical energy of a moving mass may be changed into the molecular energy of its particles, yet energy cannot be destroyed, and the energy of the universe is constant. This, the most important of the generalisations of modern science, is usually called the Conservation of Energy.

133. If, therefore, a force act upon any body or system upon which a resistance also acts, then, if the force applied

be greater than the resistance, there will be a constant increase of velocity, the excess of the work done by the force over the work done by the resistance appearing as additional kinetic energy. If the force be less than the resistance there will be a loss of velocity. If the opposing forces be equal there will be neither gain nor loss; if the body be at rest it will remain so, if in motion it will continue in motion with a uniform velocity.

Hence, if a body be at rest or in uniform motion under the action of equal and opposite forces the works done by the forces are equal and opposite. Calling the work done by one force positive, and that by the other negative, it follows that in the case of equilibrium the sum of the works done is zero.

The same holds true for any number of forces acting upon any system or machine, and the statement of this proposition is the Principle of Work or the Principle of Constancy of Work done. It is as follows:—

If any machine or system be acted upon by forces which are in equilibrium, and if the system or machine be subjected to any displacement consistent with the arrangement or connexion of its parts, the sum of the works done by all the forces is zero.

And, conversely, if the work done be zero, the forces are in equilibrium.

134. The Principle of Work, or as it has also in a slightly modified form been called the Principle of Virtual Velocities, is of course only another form of the Principle of the Conservation of Energy. And this principle is really contained in Newton's Third Law of Motion. The terms *action* and *reaction* in the Third Law may be understood in *three* distinct senses: (1) forces, (2) quantities of motion, (3) amounts of energy. The first two meanings were expressly assigned to these terms by Newton. If one body press another the *force* of action is equal to that of reaction. If one impinge on another the *quantity of motion* lost by one is gained by the other. It has been shown by Professors Thomson and Tait that Newton in a scholium to his Laws of Motion assigns to the term action a further sense, which is equivalent to that denoted by the modern scientific use

of the term *energy*, and that thus the Third Law contains within it the great principle of the Conservation of Energy.

135. Hence if a force P act on a machine upon which an opposing force R also acts, and if p denote the distance through which P moves, and r the distance through which R moves, then by the Principle of Work the sum of the works done is zero, and the work done by one must be equal and opposite to that done by the other, and therefore

$$Pp = Rr.$$

Hence there is no creation of energy with a machine. The work done by the applied power P is exactly the same as that done by R. If P be less than R, p must be greater than r. If a smaller force equilibrate a greater it can only do so by moving through a proportionally greater distance. Hence the practical rule, "what is gained in power is lost in time." It is therefore impossible to get more energy from a machine than what is applied to it. A machine can never be constructed to do work without an equal expenditure of applied work; and as in every machine, owing to the friction of its various parts, there is a transference of mechanical energy into molecular energy, it follows that the problem of "perpetual motion" is impossible.

136. If F denote the whole friction of the parts of a machine, and f the sum of the distances moved by the rubbing surfaces against friction, the equation of the preceding Art. must be written

$$Pp = Rr + Ff.$$

Where Pp denotes the work applied, Rr the effective work done, and Ff the loss of mechanical energy, or the work done against friction.

Hence the effective work obtained from a machine is never equal to the work applied. In any machine the ratio of the effective work to the applied work is called the *modulus* of the machine.

137. Units of Work.

The general expression for determining the work done by a force having been found, a *unit of work* has to be chosen. Since the work done may be estimated by a weight raised to a height (Art. 124), and since the usual unit of weight

adopted in this country is the pound weight, and the unit of length the foot, the unit of work chosen by British engineers is *the work done in raising one pound weight through a vertical height of one foot.* This unit of work is called the *foot-pound.* Hence if 20 pounds weight be raised through 5 feet, 100 units of work have been done.

In estimating the work of machines a larger unit is required, and the *rate of working* has also to be taken into account. Watt chose as his unit for this purpose the work done in raising 550 pounds weight one foot high per second, or 33,000 lbs. 1 foot high per minute. This he called a *horse-power.* Watt's estimate of the work done by a horse is now believed to be much too high, but this is of no consequence in comparing the work of machines, and his unit is that generally adopted for this purpose.

The foot-pound is the unit of work generally employed in this country by practical engineers. It is the British *gravitation unit* of work. The metrical *gravitation unit* is the centimetre-gramme. It is the work done in lifting vertically one gramme weight through the height of one centimetre.

The British *kinetic unit* of work is the *foot-poundal.* It is the work done by a poundal in moving a body through one foot. The metrical kinetic unit is the *centimetre-dyne.* It is the work done by a dyne in moving a body through one centimetre. The centimetre-dyne has received the name of the *Erg.* A foot-poundal is equivalent to 421,393·8 ergs.

138. The chief statements of the preceding Art. may be exhibited in the following form :—

$$\text{Units of Work, }\begin{cases}\text{Gravitation Units, }\begin{cases}\text{Foot-pound.}\\ \text{Centimetre-gramme}\end{cases}\\[2ex] \begin{array}{l}\text{Absolute or}\\ \quad\text{Kinetic Units,}\end{array}\begin{cases}\text{Foot-poundal.}\\ \text{Erg.}\end{cases}\end{cases}$$

Foot-pounds are reduced to foot-poundals and centimetre-grammes to ergs by multiplying by the value of g. As a mean value for this country g may be taken equal to 32·2 for the former reduction, and to 981·4 for the latter.

Examples.

1. What quantity of coal can be raised from a pit 250 feet deep by a steam engine of 56 H.P. in a day of 10 hours?

$$\frac{56 \times 33000 \times 10 \times 60}{250 \times 2240} = \text{No. of tons} = 1980 \text{ tons.}$$

2. Find the H.P. of an engine that will raise 660 cubic feet of water in an hour from a depth of 480 ft., a cubic foot of water weighing $62\frac{1}{2}$ lbs.

$$\frac{660 \times 62\frac{1}{2} \times 480}{60 \times 33000} = 10 \text{ H.P.}$$

3. What is the kinetic energy of a body weighing 128 lbs. and moving with a velocity of 100 feet per second?

$$\text{Kinetic energy} = \tfrac{1}{2} m\, v^2 = \tfrac{1}{2}\, \frac{128}{g}\, 100^2 = \frac{640000}{g}.$$

4. A cylindrical hole having a sectional area of 5 sq. feet is dug in the ground to the depth of 16 feet. If a cubic foot of earth weigh 75 lbs., what is the work done in raising the earth from the pit?

Depth of centre of gravity of the earth which is raised $= 8$.

$$5 \times 16 \times 75 \times 8 = 48{,}000 \text{ ft.-pounds.}$$

5. A weight of 20 lbs. is drawn along a table by a weight of 4 lbs. hanging vertically, the two weights being connected by a string which passes over the edge of the table. If the friction be $\frac{1}{10}$th the weight, and if after 2 seconds the string breaks, (1) how long, and (2) how far will the weight on the table move afterwards?

Acceleration

$$f = \frac{P}{P+Q}\, g = \frac{4}{24} .\ 32 = \frac{16}{3} \text{ft.} \quad \text{Velocity in 2 secs.} = f\, t = \frac{16}{3} \times 2 = \frac{32}{3}.$$

$$(1)\quad \text{time} = \frac{\text{momentum}}{\text{resistance}} = \frac{m\, v}{R} = \frac{\frac{20}{32} \times \frac{32}{3}}{\frac{20}{10}} = 3\tfrac{1}{3} \text{ secs.}$$

$$(2)\quad \text{space} = \frac{\text{kinetic energy}}{\text{resistance}} = \frac{\frac{1}{2} m\, v^2}{R} = \frac{\frac{1}{2} \times \frac{20}{32} \times \left(\frac{32}{3}\right)^2}{\frac{20}{10}} = 17\tfrac{7}{9} \text{ ft.}$$

6. The last carriage of a railway train gets loose whilst the train is running at the rate of 30 miles an hour up an incline of 1 in 150. Supposing the effect of friction upon the motion of the carriage to be equivalent to a uniformly retarding force equal to $\frac{1}{360}$ of the weight of the carriage: find (a) the length of time during which the carriage will con-

tinue running up the incline, and (*b*) the velocity with which it will be running down after the lapse of twice this interval from the instant of its getting loose.

(*a*.) 30 miles per hour = 44 ft. per sec. If W = weight of carriage, then

$$\text{Resistance} = \text{resolved part of W} + \text{friction} = \frac{W}{150} + \frac{W}{300} = \frac{W}{100}.$$

$$\text{Time} = \frac{\text{momentum}}{\text{resistance}} = \frac{mv}{R} = \frac{\frac{W}{32}.44}{\frac{W}{100}} = \frac{1100}{8} = 137{\cdot}5 \text{ secs.}$$

(*b*.) Force producing motion down the plane = resolved part of W − friction $= \frac{W}{150} - \frac{W}{300} = \frac{W}{300}$. This force produces on the mass of W an acceleration $= \frac{g}{300}$ (Art. 90). Let v = required velocity, then

$$v = \frac{g}{300}.t = \frac{32}{300} \times 137\tfrac{1}{2} = 14\tfrac{2}{3} \text{ feet.}$$

Exercises.

1. A man 12 stone weight climbs a ladder 40 feet high. Find the work done in gravitation units, and in absolute units.

2. An engine is required to raise 1,500 tons of salt per day from a mine 330 feet in depth: what should be its horse power? [Working day = 10 hrs.]

3. In what time will an engine of 30 H.P. raise 660 tons from a pit 100 ft. deep?

4. How many gravitation units of work and how many kinetic units are performed in raising 1 ton of coal from a pit 100 fathoms deep.

5. An engine of 60 H.P. is employed to raise water from a mine 50 fathoms deep: how many gallons will it raise per day of 10 hours, a gallon of water weighing 10 lbs.?

6. How many cubic feet of water will two engines, each of 25 H.P., raise in an hour from a mine 100 fathoms deep, a cubic foot of water weighing $62\frac{1}{2}$ lbs.?

7. A circular well the area of whose section is 10 feet is to be dug to a depth of 30 ft.: what will be the work of raising the material, a cubic foot of it weighing 100 lbs.?

8. A mill pond is 120 ft. long, 50 ft. broad, and is filled with water to the depth of 2 ft. If the fall of the water be 4 ft., required the potential energy of the water.

9. What is the work done per second by a fire engine which throws a cubic foot of water per second with a velocity of g feet per second?

10. How many cubic feet of water can be raised in an hour from a well 410 feet deep by an engine of 50 H.P., allowing 25 per cent. for friction, leakage, &c.?

11. At what fraction of a H.P. does a man work on the average who can do 900,000 units of work in a working day of 9 hours?

12. What H.P. steam engine will be required to raise 10 tons of coal per hour from a depth of 600 feet?

13. An engine of 15 H.P. raises 3,000 cub. feet of water per hour from a well 30 feet deep: what is the modulus of the machine?

14. Which could you throw further, a ball of lead or a ball of cork of the same size? Give the reason for your answer.

15. A weight of 31 lbs. is placed on a table and connected by a string with a weight of 1 lb. which hangs vertically and draws the weight of 31 lbs. along the table. After 3 seconds the cord breaks, and the weight on the table moves for $4\frac{1}{2}$ feet further before coming to rest: what is the friction?

16. A stone weighing 12 lbs. is thrown along ice with a velocity of 64 feet per second, and comes to rest after moving 192 yards: what is the force of friction?

17. A weight of 24 lbs. is drawn along a table by means of a cord by a weight of 2 lbs. hanging vertically. After 3 seconds the cord breaks and the weight on the table moves for 4 seconds before coming to rest: what is the friction?

18. A stationary engine draws from rest a waggon weighing 10 tons up a smooth incline of 30° with an acceleration of 4 feet per second. After 10 seconds the rope connected with the waggon breaks: how long and how far will the waggon move upwards?

19. A weight of 50 lbs. is drawn along a horizontal table by a weight of 10 lbs. hanging vertically. After 3 seconds the cord breaks, and the 50 lbs. moves for $12\frac{1}{2}$ feet before coming to rest: what is the friction?

20. Two weights of 3 lbs. and 4 lbs. are connected by a cord passing over a pulley. A bar weighing 2 lbs. is placed on the 3 lbs. weight, and after motion has continued for 9 seconds the bar is removed: for what distance further will the 3 lbs. weight descend?

21. What is the kinetic energy of a moving body? What is the *vis viva*?

22. What are the gravitation units of work usually employed? What are the kinetic units?

23. A body weighing 100 lbs. moving with a velocity of 96 feet per second is retarded by a force of 8 lbs.: find (1) how long, (2) how far the body will move.

24. A stone weighing 20 lbs. is thrown along ice with a velocity of 60 feet per second. If the friction be $\frac{1}{10}$th of the weight, find (1) how long, (2) how far the stone will move.

25. A locomotive weighing 1,000 lbs. and moving with the velocity of 48 feet per second on a level railroad is brought to rest by the brake in a space of 200 feet: what is the force of the brake?

26. A train weighing 60 tons is moving on a horizontal line at the uniform rate of 15 miles per hour. If the resistance of the air and friction be 20 lbs. per ton, what is the horse power of the engine?

27. If in the foregoing case the train were moving up an incline of 1 in 60, what is the H.P.?

28. A railway train weighing 200 tons is running at the uniform rate of 20 miles per hour when the steam is shut off. The friction and other resistances being taken as 8 lbs. per ton, find how far the train will move before coming to rest.

29. A train weighing 60 tons has a velocity of 40 miles an hour: if the steam be turned off, how far will it ascend an incline rising 1 foot in 100, the resistance from friction, &c., being 8 lbs per ton?

30. What is the horse-power of an engine capable of drawing a train, which with the engine weighs 100 tons, up an incline of 1 in 50 with a velocity of 30 miles per hour, the friction and other resistances being 14 lbs. per ton?

31. A stationary engine draws a railway waggon weighing 10 cwts. up an incline of 1 in 56 with an acceleration of 20 feet per second. After 6 seconds from rest the rope connected with the waggon breaks. If the friction be equivalent to a uniform force of $\frac{1}{100}$th of the weight of the waggon, find (*a*) how long the waggon will move up the incline, and (*b*) the velocity with which it will be running down after the lapse of twice this interval from the instant of the breaking of the rope.

32. If in the foregoing case the incline were smooth, find (*a*) how long motion will continue upwards after the rope breaks, and (*b*) the velocity after twice this interval.

33. A ton weight is raised through a height of 10 feet: what is the work done in poundals, g being 32?

34. Express in foot-poundals the kinetic energy of a body weighing 100 lbs., and moving with a velocity of 20 feet per second.

35. A weight of 500 grammes is raised through a vertical height of 5 metres in a place where g is 980: find the work done (*a*) in centimetre-grammes, (*b*) in ergs.

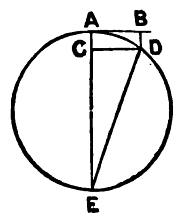
36. What is the kinetic energy, expressed in ergs, of a kilogramme which is moving with a velocity of 1 metre per second?

37. If an engine working at the rate of 30 H.P. draw a train which, with the engine, weighs 120 tons, along a horizontal railroad, what uniform velocity will it maintain when the resistances from friction and the air are 10 lbs. per ton?

CHAPTER X.

MOTION IN A CIRCULAR PATH WITH UNIFORM VELOCITY.

139. It was shown in Chapter V. that if a body possess a uniform velocity in one direction and be acted upon by a constant force in another, the body will move in a curved path, the nature of the curve depending upon the direction and magnitude of the force. We are now about to investigate the direction and magnitude of the force when the body describes a circle.



140. Let a body of mass m be moving with uniform velocity in the circumference of a circle whose radius is r. Then some force must act on the body, otherwise, by the First Law of Motion, it would move in a straight line. And since the body is moving uniformly, this force neither accelerates nor retards the velocity, and consequently the force must always act at right angles to the path of the body, and must therefore be constantly directed to the centre of the circular path in which the body is moving. To find the magnitude of this force:—Let F be the force and f its acceleration. Let the body of mass m be at A and be moving with uniform velocity v; then if no force acted on the body it would move in a straight line along AB. Let t be the time in which it would reach B. During this time a force F directed towards the centre acts constantly on it. Let AC be the space which it would describe under the influence of this force in the same time. Then completing the parallelogram, the body will at the end of the time t be found at D (Art. 52) having moved along the arc AD.

If the time t be very small AD may be considered a straight line, and BD directed towards the centre; and since AD is described with a uniform velocity v in the time t $\therefore$ (Art. 6),

$$AD = vt.$$

$$\text{and } AC = \tfrac{1}{2}ft^2 \text{ (Art. 12).}$$

$$\text{But (Euc. VI. 8) } EA \cdot AC = AD^2.$$

$$\therefore 2r \cdot \tfrac{1}{2}ft^2 = v^2t^2,$$

$$\therefore f = \frac{v^2}{r} \qquad (1)$$

$$\text{But (Art. 75) } F = mf,$$

$$\therefore F = \frac{mv^2}{r}. \qquad (2)$$

141. Hence if a body of mass m possess a velocity v, it will move in the circumference of a circle whose radius is r if it be acted on by a force $\frac{mv^2}{r}$ constantly directed towards the centre of the circle. And conversely, if a body move uniformly with the velocity v in the circumference of a circle whose radius is r, the force acting on the body (or the resultant of all the forces acting on it) is equal to $\frac{mv^2}{r}$, and is constantly directed to the centre of the circle.

Thus the force required to keep a body of given mass moving in the circumference of a circle is directly proportional to the square of the velocity, if the radius be given, and is inversely proportional to the radius, if the velocity be given.

This impressed force is *centripetal*, that is directed towards the centre.

142. If the force cease to act, the body continues its motion in a straight line in the direction in which it was moving at that instant, that is in the tangent to the circle. Thus if a stone tied to one end of a string, the other end of which is held in the hand, be whirled in a circle with uniform velocity, the string has to be kept pulled with a constant force directed always from the stone to the hand. If the

string break, the stone will fly off, not in a direction outward from the hand, but in a tangent to the circular path in which it is moving. But whenever force is exerted on a body there is always a reaction equal and opposite to the action. The stone whirled in a circle has a constant tendency, owing to its inertia, to move in the tangential path. To overcome this inertia, a constant pull has to be maintained, which is transmitted by the string, and this causes an equal reaction upon the hand and in the opposite direction. This strain or reaction is called the *centrifugal force*. It is equal and opposite to the *centripetal*.

It should be borne in mind that this centrifugal force is not an external impressed force. It is the reaction due to the impressed force, ceasing when it ceases, and always equal to it in magnitude but opposite in direction.

143. It should also be observed that in the case of the stone whirled in a circle, no *tangential force* acts *during* the uniform circular motion. The body acquires its velocity by some force which then ceases to act, and it would continue to move in a straight line with this velocity if no force acted on it. The centripetal force changes the rectilinear into circular motion.

144. Equations (1) and (2) of Art. 140 may be put into other forms which are more convenient for some calculations.

Let T denote the *periodic time*, that is the time taken by the body to make one revolution, and let π denote as usual the ratio of the circumference of a circle to the diameter, which is approximately 3·14159. Then if r be the radius, the circumference is $2\pi r$; and as this is described with the uniform velocity v in the time T, by Art. 6,

$$v = \frac{2\pi r}{T},$$

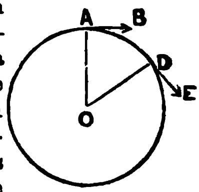
Substituting in equations (1) and (2) we obtain

$$f = \frac{4\pi^2 r}{T^2}. \qquad (3)$$

$$F = m\frac{4\pi^2 r}{T^2}. \qquad (4)$$

Thus the centripetal or the centrifugal force is directly proportional to the mass, directly proportional to the radius, and inversely proportional to the square of the periodic time.

145. *Angular Velocity.*—Let, as in Art. 140, a body be moving with a uniform velocity v in the circumference of a circle whose radius is r. Then since the body moves uniformly, the radius drawn from the centre to the body in the circumference moves through equal angles in equal times, and the rate at which the radius rotates is called the *angular velocity* of the body.



Let the body arrive at the point D; the angle between the directions of the tangents AB and DE, which are the directions of the motion of the body at these points, is equal to the angle AOD: thus the angular velocity may be called the *rate of change of direction of motion.* The angular velocity is usually denoted by the letter ω. If, as in Art. 140, T be the periodic time, and since 2π is the circular measure of 360°, it follows that $\omega = \frac{2\pi}{T}$. Substituting this in equation (4) of Art. 144, we obtain

$$F = m\omega^2 r.$$

An equation which expresses the centrifugal or centripetal force in terms of the mass of the body, the angular velocity, and the radius of the circle described.

146. Numerous familiar illustrations of the foregoing principles may be adduced.

The mud from the wheels of a carriage in motion is seen to be thrown off in tangents to the wheel. A bucket full of water may be whirled in a vertical plane, the centrifugal force counterbalancing the weight of the water. In a circus when a horse walks round the ring, both horse and rider are upright, but when he gallops in the circle both lean inwards, and the quicker the motion the more they incline. The centrifugal force in this case is balanced by a portion of the

weight of the horse and the rider, and the greater the centrifugal force the greater must be the portion of the weight applied to counteract this force, and consequently the greater must be the incline.

147. The centrifugal force due to the earth's rotation produces two distinct effects which we shall now consider.

Owing to the rotation of the earth on its axis all bodies on its surface, except at the poles, have a tendency to fly off in tangents to the circles they are describing. A part of the force of gravity acting on each body is applied to prevent this motion, and thus the weight of the body and the acceleration due to gravity are less than they would be if the earth were at rest. Hence, the whole acceleration due to the force of gravity is greater than that observed by the acceleration due to the centrifugal force. Calling the whole acceleration G, the observed acceleration g, and the acceleration due to the centrifugal force f, then

$$G = g + f.$$

In the case of a body at the equator g has been determined by experiment to be about 32·088 feet per second; and since the velocity of a body at the equator, and the radius of the circle described are known, f may be shown by equation (1), Art. 140, to be about 0·1113 feet per second, and, hence, $g = 288{\cdot}4 f$, and, therefore,

$$G = 289{\cdot}4 f.$$

That is, the force of gravity is about 289 times the centrifugal force at the equator. Therefore, since the centrifugal force is proportional to the square of the velocity, if the earth's rotation were 17 times more rapid, the centrifugal force at the equator would be equal to the force of gravity, and bodies at the equator would have no weight. With a still more rapid rotation they would fly off in tangents to the equator.

148. The centre of the circle described by a body at the equator is the centre of the earth, and, hence, the centrifugal force which acts from the centre is approximately opposite in direction to the force of gravity. At any place

intermediate between the pole and equator a body describes during the rotation of the earth the parallel of latitude of the place. The centre of this parallel is not the earth's centre, and, hence, the direction of the centrifugal force is not that of gravity. Resolving the centrifugal force in the direction of gravity and at right angles, we obtain two components, one of which acts in opposition to gravity, and the other tends to move the body towards the equator. Thus the centrifugal force at any parallel has two effects. It lessens the weights of bodies and the acceleration due to gravity, and it tends to move these bodies towards the equator.

It is evident that this diminution of the weights of bodies owing to the earth's rotation is greatest at the equator and least at the poles, where there is no centrifugal force.

Hence, if the earth were originally a spherical molten mass, it would by its rotation become an oblate spheroid.

EXAMPLE.

1. One end of a string 2 feet long is fastened to a point in a smooth horizontal table, and to the other end is attached a weight of 4 lbs., which is caused to revolve round the fixed point as centre with a uniform velocity of 4 feet per second: find the tension of the string.

$$F = \frac{mv^2}{r} = \frac{\frac{4}{32} \times 4^2}{2} = 1 \text{ lb.}$$

EXERCISES.

1. A stone is tied to the end of a string 4 feet long, the other end of which is held in the hand, and whirled round in a vertical plane. What must be its velocity at the highest point of the circle that the string may just remain stretched?

2. A tram car weighing 1 ton and moving at the rate of 3 miles an hour passes round a curve whose radius is 132 yards: what is the outward pressure on the rails?

3. A weight of 4 lbs. attached to the end of a string 2 feet long, the other end of which is fixed to a point in a smooth horizontal table, moves on the table in a circular path round the fixed point as centre, and describes the circumference every 3 seconds: find the tension of the string.

4. If in the foregoing case the string can just support a weight of 36 lbs.: what is the greatest velocity that can be given to the weight before the string breaks?

5. A railway train is moving with the uniform velocity of 30 miles per hour along a curve in the rails, the radius of which is 500 yards. From the roof of one of the carriages a weight of 12 lbs. is suspended by a string: what horizontal force should be applied to this weight so as to keep the string vertical while the train is rounding the curve?

6. A body weighing 8 lbs. is moving uniformly in a circle whose radius is 10 feet, and makes a revolution every 4 seconds: what is the force acting on the body?

7. If a body be moving in a circular path with uniform velocity, what inferences may be drawn respecting the forces acting on the body?

8. Explain clearly what is meant by the so-called *centrifugal force*.

9. What are the effects of the centrifugal force, due to the rotation of the earth, on bodies (*a*) at the equator, (*b*) at the poles, (*c*) at places intermediate between the pole and equator?

10. Prove that the attraction of gravity is 289 times the centrifugal force at the equator due to the earth's rotation.

11. A stone weighing 1 lb. is tied to one end of a string 4 feet long, the other end being held in the hand, and the stone is whirled in a horizontal circle. If the string can just bear a tension of 2 lbs., with what velocity is the stone moving when the string breaks?

12. A body whose mass is 5 lbs. moves with a uniform velocity of 80 feet per second in a circle whose radius is 10 feet. What force must act upon it, and in what direction?

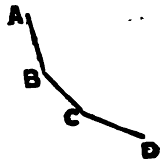
13. A locomotive weighing 10 tons, and moving at the rate of 15 miles per hour, passes round a curve which is a portion of a circle whose radius is 1,000 feet: find the centrifugal force.

CHAPTER XI.

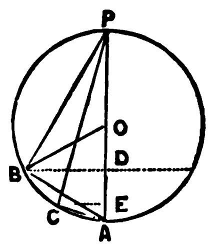
MOTION IN A CIRCULAR ARC WITH VARIABLE VELOCITY.

149. We shall now consider the case of a body falling under the influence of gravity down a circular arc in a vertical plane, and we shall investigate (1) the *velocity* acquired by a body in falling down any circular arc in a vertical plane, and (2) the *time* taken by a body to describe an arc in a vertical plane when the middle point of the arc is the lowest point.

150. *The velocity acquired by a body in falling down a circular arc in a vertical plane is equal to what it would acquire in falling through the vertical height of the arc.*



Let AB, BC, CD, ... be a series of planes whose vertical heights are $h_1, h_2, h_3,$ . . . . h_n. Let a body fall down these planes in order without any loss of velocity in passing from plane to plane, and let the velocities at B, C, D . . . . be respectively $v_1, v_2, v_3,$ . . . . v_n. Then (Art. 14,) $v_1^2 = 2gh_1$; $v_2^2 = v_1^2 + 2gh_2$; $v_3^2 = v_2^2 + 2gh_3$, $v_n^2 = v_{n-1}^2 + 2gh_n$. Adding these equations we obtain $v_n^2 = 2g(h_1 + h_2 + h_3 + . . . h_n) = 2gh$ where $h =$ the whole vertical height.



Let BC be an arc of a circle whose plane is vertical. Then BC may be regarded as formed of an infinite number of inclined planes; and it may be shown that when a body falls down such a system of planes, there is no loss of velocity. Hence if the velocity in falling down BC be called v, then by the preceding demonstration $v = \sqrt{2g.\text{DE}}$. Therefore (Art. 44,) the velocity is the same as that acquired in falling through the vertical height DE.

151. The velocity acquired in falling down any arc of a circle in a vertical plane may be expressed in terms of the chords drawn from the extremities of the arcs to the lowest point of the circle.

Let OA (Art. 150) $= l$, and PA $= 2l$. Let the chord BA $= a$, and CA $= x$. Then (Euc. VI., 8)—

$$\text{DA} = \frac{\text{AB}^2}{\text{PA}} = \frac{a^2}{2l}, \text{ and EA} = \frac{\text{AC}^2}{\text{AP}} = \frac{x^2}{2l}.$$

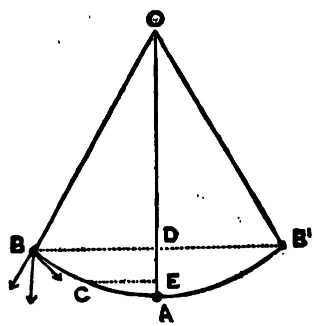
$$\text{But (Art. 150) } v = \sqrt{2g\,\text{DE}} = \sqrt{2g\,(\text{DA} - \text{EA})} =$$

$$\sqrt{2g\left(\frac{a^2}{2l} - \frac{x^2}{2l}\right)} = \sqrt{\frac{g}{l}(a^2 - x^2)}.$$

Similarly if V represent the velocity at the point A after falling through the arc BA,

$$\text{V} = \sqrt{2g.\,\text{DA}} = \sqrt{2g\frac{a^2}{2l}} = a\sqrt{\frac{g}{l}}.$$

152. *The Simple Pendulum.*—Let a material particle be suspended from the point O by a weightless string, and let it be drawn aside to any point B. Then the weight of the particle acting vertically downwards may be resolved into two components, one in the direction OB and the other at right angles. The former is met by the reaction of the fixed



point, the latter gives motion to the particle. Owing to the string the particle is forced to move in an arc of a circle whose radius is OA, and it will be readily seen that the component causing motion diminishes from B to A where it is zero. The motion is therefore similar to that of a body falling down a circular arc, and the velocity at any point C can be determined exactly as in Art. 151. The particle when it has reached A has its greatest velocity. At this

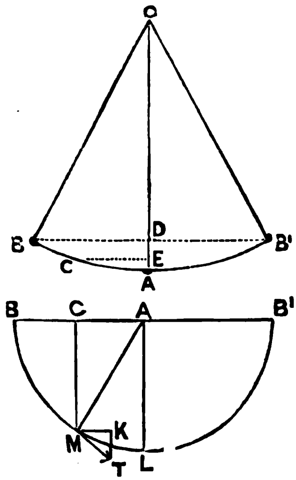
point it possesses an amount of kinetic energy which, if none of it were lost, would carry it up to B′ at the same height as B. Here it would come to rest for an instant and it would then fall again towards A, rise to B, and thus continue oscillating between B and B′. Practically, however, owing to the resistance of the air and the imperfect fixedness of the point of support, there is a constant loss of energy and the body after a time comes to rest.

Such a material particle suspended by a weightless string or rod and oscillating in a vertical plane round a fixed point is called a Simple Pendulum. The radius OA is the *length* of the pendulum, the arc BB′ is the *range* or *amplitude of the vibration*, and the time taken to describe BB′ is the *time of vibration.*

A simple pendulum cannot be actually constructed, but a close approximation to it may be made. A very small lead ball suspended by a fine silk string may for most experiments be regarded as a simple pendulum. A Compound Pendulum is a mass of any shape oscillating round a fixed axis.

153. When the ranges are very small compared with the length of the pendulum, the time of a vibration does not depend on the length of the range. Galileo was the first to observe this fact, and to apply the pendulum to measure equal intervals of time. It becomes of great importance therefore to determine on what the time of vibration of a pendulum depends and we proceed to find an expression for the time of vibration of a Simple Pendulum when oscillating in small arcs.

[*] 154. *To find the time of vibration of a simple pendulum.* Let A be a material particle suspended from O by a string OA, supposed to be without weight, and let it oscillate in an arc BB′ so small that the arcs BA, CA may be regarded as coinciding with their chords. Let $OA = l$, $BA = a$, $CA = x$. At any point C let the velocity be called v, and at A be V; then (Art. 152) the velocity at B is zero; the velocity v at $C = \sqrt{2g \cdot DE} = \sqrt{\frac{g}{l}(a^2 - x^2}$; and the velocity V at $A = \sqrt{2g \cdot DA} = a\sqrt{\frac{g}{l}}$. Let BCAB′ be a



straight line equal to the arc BB', and let a particle oscillate on this straight line with the same velocities as the pendulum on the arc BB'. Each will therefore start from the point B with the velocity o, will have a velocity v at C and a velocity V at A, and equal velocities at the corresponding points of their paths. Upon BB' describe a semicircle BMLB', and let another particle move along this semicircular path with the uniform velocity $V = a\sqrt{\frac{g}{l}}$. Then the time taken by this latter particle to describe the semicircumference will be the same as that taken by the former to describe the straight line BB'. For the horizontal velocity of the particle moving on the semicircumference is at any instant the same as the velocity of the particle moving on BB'. At the point C draw a vertical CM. Suppose the particle moving on the semicircumference is at M; its velocity being uniform is $a\sqrt{\frac{g}{l}}$.

Resolve this horizontally, and we obtain $a\sqrt{\frac{g}{l}}.\ \cos \mathrm{TMK} =$

$$a\sqrt{\frac{g}{l}}.\ \cos \mathrm{AMC} = a\sqrt{\frac{g}{l}}.\ \frac{\mathrm{CM}}{\mathrm{AM}} = a\sqrt{\frac{g}{l}}\frac{\sqrt{\mathrm{AM}^2 - \mathrm{AC}^2}}{\mathrm{AM}}$$

$$= a\sqrt{\frac{g}{l}}\ .\ \frac{\sqrt{a^2 - x^2}}{a} = \sqrt{\frac{g}{l}\left(a^2 - x^2\right)} = v.$$

Therefore the horizontal velocity of the particle at M is equal to the velocity of the particle at C.

Similarly it may be shown that the velocities at L and A are each $a\sqrt{\frac{g}{l}}$, while at B' both velocities are zero.

Hence the particles will move throughout on the extremities of the same vertical line, and therefore the time taken by each to move from B to B′ will be the same.

But the semicircumference is equal to πa, where $\pi = 3{\cdot}14159$, and $a =$ the radius $=$ BA. And since this distance is described with uniform velocity the time taken is, (Art. 6),

$$= \pi a \div a\sqrt{\frac{g}{l}} = \pi\sqrt{\frac{l}{g}}.$$

This is also the time taken to describe the line BB′ and is therefore the time of a vibration of the pendulum. Let this time be called t, then

$$t = \pi\sqrt{\frac{l}{g}}.$$

155. From this formula it appears that the time of a vibration depends only upon the length of the pendulum and the force of gravity at the place in which it is set vibrating. In the same place, the times of vibration of different pendulums are therefore directly proportional to the square roots of the lengths; and in different places, the times of vibration of the same pendulum are inversely proportional to the square roots of the numbers which express the accelerations produced by gravity. We learn also from the formula that the time of vibration is not influenced by the weight or the nature of the substance of the oscillating body—since these do not enter into the formula—and thus we have another confirmation of the fact mentioned in Art. 77 that the force of gravity does not depend upon the nature of the mass but only upon its amount.

It will be seen that by the *time of a vibration* is meant the time taken to swing from one extremity of the arc to the other. This is strictly only a semi-oscillation of the pendulum. The time of a complete oscillation is the time taken to move from one extremity of the arc to the other and back again, and is therefore double the value given in Art. 154.

156. All of the foregoing statements may be verified by experiment. If small masses differing in weight and in material be suspended by fine strings of the same length

and set oscillating in small arcs, it will be found that their times of vibration are practically the same; but if the pendulums be of different lengths their times are found to be proportional to the square roots of their lengths.

157. By squaring the equation of Art. 154, and bearing in mind that the number of vibrations in a given time is inversely proportional to the time of vibration we deduce the following laws of the vibration of the Simple Pendulum :—

(1.) With different pendulums in the same place, *the lengths are directly proportional to the squares of the times of vibration.*

(2.) With the same pendulum in different places, *the values of the accelerations produced by gravity are inversely proportional to the squares of the times of vibration and directly proportional to the squares of the numbers of vibrations in the same time.*

158. The equation of Art. 154 is rigorously true only when the arc of vibration is indefinitely small. A modified form of the formula may, however, be obtained which is applicable to any amplitude of vibration. But, further, this formula is applicable only to the *simple pendulum*, which can never be absolutely realised; and as an ordinary pendulum consists of a large mass called the bob suspended by a rod or wire whose mass cannot be neglected, it becomes of importance to learn how the formula for the simple pendulum may be applied to the ordinary form of the pendulum. A *compound pendulum* is made up of particles at different distances from the point of suspension. These particles if free to move separately would swing at different rates of vibration, but being connected together the whole mass oscillates in a way determined by the form of the vibrating body. Now, it can be shown by theory—but the demonstration cannot be given here—that there is a point in every compound pendulum, called the *centre of oscillation*, at which if the whole mass of the oscillating body could be concentrated the time of vibration would not be altered. The vibrations of any compound pendulum are therefore the same as those of a simple pendulum whose length is the distance between the centres of suspension and oscillation. By calling this

distance the *length* of the compound pendulum, the formula for the simple pendulum may be applied to the compound.

159. It may be further shown that if the centre of oscillation of any vibrating mass be found, and if it be made the point of suspension the former point of suspension will be the new centre of oscillation. Thus the centres of suspension and oscillation are convertible.

160. The *length* of a compound pendulum, that is the length of the equivalent simple pendulum, may be approximately determined by experiment. A very small lead bullet suspended by a fine silk string may be regarded as a simple pendulum. This is set oscillating on the same axis as the compound pendulum, and the string is shortened or lengthened until the vibrations coincide with those of the compound pendulum. The distance from the axis to the centre of the bullet is then very approximately the length of the compound pendulum.

A still more accurate method is based upon the convertibility of the centres of suspension and oscillation. The body is made to oscillate round any axis of suspension, and the time of vibration determined. Then by repeated trials another axis is found around which the body oscillates in the same time. The distance between the two axes is the length of the equivalent simple pendulum. Captain Kater employed a modification of this method. In his pendulum the two axes are fixed and a small weight can be moved between them and clamped at any point. The position of the centre of oscillation depends upon the arrangement of the mass of the oscillating body, and by repeated trials the movable weight is so placed that the centre of oscillation is made to coincide with one of the axes when the other is the axis of suspension. This is known when the time of vibration round each axis is the same, and the length of the pendulum is the distance between the axes.

161. In the equation $t = \pi\sqrt{\frac{l}{g}}$, any two of the three variable quantities being known the remaining one may be calculated. Hence since in the case of any pendulum the length and the time of vibration can be determined, the

value of g at any place may be calculated. This is the most accurate method for determining the acceleration produced by gravity.

The length of the pendulum is found with great precision by Kater's method, and the time of vibration may be found by the method of coincidences, as follows:—

The pendulum whose time of vibration we wish to determine is placed exactly in front of the pendulum of an astronomical clock beating seconds, and both are viewed from a distance through a telescope. When at rest the two pendulum rods coincide in the centre of the field of view. Both are now drawn aside to the same distance and let go at the same moment. They cross the field of view together, and the instant at which they do so is given by the clock. Let us suppose that the experimental pendulum moves somewhat more quickly than the clock pendulum. The former will soon be observed crossing in advance of the latter, the distance between them at each vibration gradually increasing until the experimental pendulum gains half a vibration on the other. The pendulums now cross in opposite directions, and the distance between them gradually diminishes until they are seen to coincide, but moving in opposite directions. The experimental pendulum has now gained one vibration on the clock pendulum. The experiment is usually continued until another coincidence takes place when *two* vibrations have been gained, and the pendulums are now moving in the same direction as when started. The time is noted at that instant on the clock. Suppose the interval between the two coincidences is 500 seconds, then the clock pendulum has made 500 vibrations, and the experimental 502; and the time of a single vibration of the latter is therefore $\frac{500}{502}$ sec.

This method enables us to determine the time of a vibration with great ease and precision. The clock registers the time between the coincidences, and any error in estimating this time becomes inappreciable when divided among so many vibrations.

The length of the experimental pendulum and the time of

vibration at any place having been thus determined, the value of g at the place is known from the formula of Art. 154,

$$g = \frac{\pi^2 l}{t^2}.$$

162. The value of g has thus been determined very accurately in a great number of places over the earth. In London g has been found to be 32·1908 feet per second.

The value of g varies with the latitude. This is due to two causes. The earth is an oblate spheroid, and therefore its surface near the equator is further from the centre than that near the poles; and as the force of the attraction of gravitation varies inversely as the square of the distance from the centre, the force of gravity must increase from the equator to the poles. Again it has been shown in Art. 148 that owing to the action of the centrifugal force due to the earth's rotation the force of gravity also increases from the equator to the poles. Therefore for both reasons the acceleration due to gravity increases with the latitude.

163. The value of the acceleration of gravity diminishes with the altitude. If g be the value at the level of the sea and g' the value at a height h, and if R denote the radius of the earth, then according to the law of inverse squares

$$g' : g :: R^2 : (R+h)^2.$$

Thus if the value of the acceleration be found by experiment at any height, the value at the sea level may be calculated.

164. In equation of Art. 154 by putting $t = 1$, $l = \frac{g}{\pi^2}$. Thus the length of the seconds pendulum at any place depends on the value of g at the place and can be calculated if g be known. In London the length of the pendulum which makes a vibration in 1 second has been found to be 39·1393 inches. The length at any place varies with the value of g, and therefore the seconds pendulum must be shortened as we go towards the equator and lengthened as we go towards the poles.

The variation in the rate of a pendulum due to change of

place or to change of the pendulum's length, can be readily calculated by the principles of Art. 157 when we know the value of g.

EXAMPLES.

1. In what time will a pendulum a mile long make one vibration, where $g=32$?

$$t=\pi\sqrt{\frac{l}{g}}=\pi\sqrt{\frac{5280}{32}}=3\cdot14159\sqrt{165}\text{ secs.}$$

2. A pendulum which beats seconds in London where $g=32\cdot19$ is carried to another place where it makes 100 oscillations more in a day: what is the value of the acceleration due to gravity in the latter place?

Let g' = its value.

The number of oscillations made by the seconds pendulum in 1 day in London = 86,400.

$$g' : g :: 86500^2 : 86400^2 \therefore g'=\frac{86500^2 g}{86400^2}.$$

3. The length of the seconds pendulum in a certain place is 39·1393 inches: what is the value of g?

$$g=\frac{\pi^2 l}{t^2}=\frac{\pi^2 l}{1^2}=\pi^2 l=\frac{(3\cdot14159)^2\times39\cdot1393}{12}\text{ feet}=32\cdot19\text{ feet.}$$

EXERCISES.

1. What is the length of the seconds pendulum in London, if $g=32\cdot1908$?

2. If the length of a pendulum which beats seconds be 39·1393 inches what is the length of a pendulum which beats quarter seconds?

3. A pendulum which beats seconds at the equator would if carried to the pole gain 5 minutes a day: compare the forces of gravity at the equator and the pole?

4. The pendulum of a clock beats seconds and is 39·1393 inches long. If it be shortened by the $\frac{1}{10}$th of an inch, how many seconds will the clock gain in a day?

5. A clock having a seconds pendulum is carried to the top of a mountain 3 miles high: how many seconds will the clock lose in a day, the radius of the earth being taken as 4,000 miles?

6. The lengths of the seconds pendulum in two different places are 39·15 and 39·14 inches respectively: compare the force of gravity in the first place with that in the second.

7. If the length of the seconds pendulum in Dublin be 39·15 inches, what length of pendulum would vibrate once in a minute?

8. If the value of g in Dublin be 32·2 what will be the length of the seconds pendulum, the length in London being 39·1393 inches, and $g=32\cdot19$?

9. A clock whose pendulum beats seconds in London is carried to the equator: how many seconds will it lose in a day? (g is equal to 32·19 in London, and 32·09 at the equator.)

CHAPTER XII.

DYNAMICS—STATICS.

FORCE AND REST.

Composition and Resolution of Forces acting on a Particle.

165. It has been stated in Art. 105 that when any number of forces act upon a body, each produces an acceleration proportional to itself in magnitude and in its own direction. These accelerations may in certain cases balance each other, and the body will consequently remain at rest. In such cases the forces always produce their full effects, and it is really the *effects* of the forces that neutralise one another. It is, however, usual to say that the forces themselves equilibrate, and to treat of the equilibrium of forces under that branch of Dynamics called Statics. If forces acting on any body be in equilibrium, they will be in equilibrium if applied to any other body, whatever the mass may be. Hence, in statical problems it will not be necessary to take into account the *mass* of the body upon which the forces act.

In this chapter we shall consider forces acting on a *particle.*

166. *Composition of Forces acting on a Particle.*—A particle may be acted on at the same time by two or more forces in the same or in different directions, and some one force may be found whose effect is equal to the combined effect of these forces. The forces acting at the same instant on the particle are called the Components, and the force equal to their combined effect is called the Resultant, and the determination of the resultant when the component forces are given is called the Composition of Forces.

In the Composition of Forces we make use of the principle of Art. 105, and of the demonstrations in chapter III. under Composition of Velocities.

167. *Composition of Forces acting on a Particle in the same straight line.*

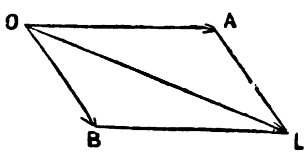
Since (Art. 105) if the particle were free to move, each force acting on it would produce in the unit of time a velocity proportional to itself in magnitude and in the same direction, and since the resultant velocity would be proportional to the resultant force in magnitude and direction, it follows from Art. 21 that if any number of forces act on a particle in the same straight line, then calling those acting in one direction positive, and those in the other negative, the resultant is the algebraic sum of the forces.

168. Hence, there cannot be equilibrium when only one force acts on a particle. If two forces act on a particle there is equilibrium only when the forces are equal and opposite.

169. *Composition of two Forces acting on a Particle in directions which are not in the same straight line.*

If two forces act on a particle in directions which are inclined to each other at a given angle the resultant may be found by the following proposition :—

Parallelogram of Forces.—*If two forces acting on a particle be represented in magnitude and direction by two adjacent sides of a parallelogram, the diagonal of the parallelogram drawn through the particle will represent the resultant force both in magnitude and direction.*



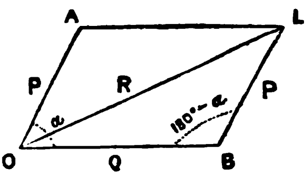
Let two forces represented by the lines OA and OB act on the particle O, then OL represents the resultant of these forces. By Art. 105 each force will produce a velocity in the unit of time proportional to itself in magnitude and direction. Therefore the lines OA and OB also represent the velocities in the unit of time due to the two forces. But (Art. 22) the resultant of these

velocities is represented by the line OL, and (Art. 105) the resultant force is proportional to the resultant velocity, therefore the line OL also represents both in magnitude and direction the resultant of the forces represented by OA and OB.

Thus the resultant of two forces OA and OB acting on a particle in directions inclined at the angle AOB is determined geometrically by completing the parallelogram and drawing the diagonal OL.

170. If the numerical values of the forces represented by OA and OB be given, the resultant OL may be calculated precisely as in Art. 26 without the aid of trigonometry, when the angle AOB is any one of the following:—30°, 45°, 60°, 90°, 120°, 135°, 150°.

171. A general formula may be found, as in Art. 27, by which the resultant of two forces whose directions are inclined at a given angle may be calculated.

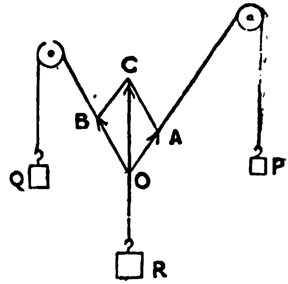


Let two forces P and Q act on the particle at O in directions inclined at an angle AOB $= \alpha$. Take OA, OB proportional to P and Q respectively, complete the parallelogram, and draw OL. Let OL = R. Then by trigonometry, as in Art. 27,

$$OL^2 = BL^2 + OB^2 - 2BL \, . \, OB \cos(180^\circ - \alpha)$$
$$\therefore R^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

This is the general formula required.

172. From the foregoing demonstration it is seen that the greater the angle between the directions of the forces the less the resultant, and that the direction of the resultant is always nearer the greater force. The determination of the angles which the direction of the resultant makes with the directions of the components is the same as the trigonometrical problem, *given the sides of a triangle to find the angles.*



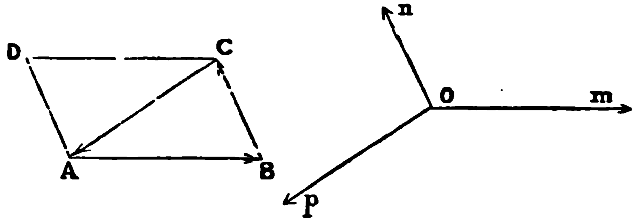
Experimental verification of the Parallelogram of Forces.—The Parallelogram of Forces may be verified as follows:—Three fine strings are knotted together at a point O, and weights P, Q, R are attached one to each extremity. Two of the strings are passed over pulleys fixed on a vertical board and turning freely in a vertical plane, the other hangs vertically. The point O is acted upon by three forces P, Q, R; and since it is in equilibrium, the combined effect of P and Q upwards is just balanced by R downwards. R, therefore, is equal and opposite to the resultant of P and Q. If, now, OA and OB be measured off proportional to P and Q respectively, and the parallelogram OACB be completed, then OC will be found to be proportional to R and in the vertical direction in which R acts. This result will always be found, no matter how P, Q, R are varied, provided that equilibrium is possible with the weights selected. Hence, since R is equal and opposite to the resultant of P and Q, and since OC represents R in magnitude and is opposite in direction, therefore OC the diagonal of the parallelogram represents the resultant of the forces which are represented by the sides OA, OB.

174. Triangle of Forces.—*If a particle be acted on by three forces which are represented in magnitude and direction by the sides of a triangle taken in order, then the forces will be in equilibrium.*

Conversely.—*If three forces acting on a particle be in equilibrium, and if any triangle be drawn whose sides are parallel to the directions of the forces, the sides of the triangle will represent the forces in magnitude.*

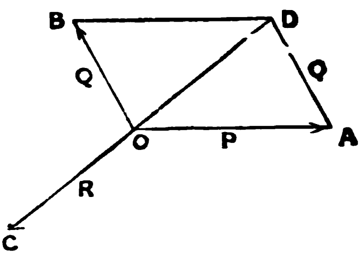
Let AB, BC, CA represent forces acting on a particle. Then (Art. 169) the resultant of AB and BC is AC. Substituting the resultant for the components, the forces acting

on the particle are equivalent to the two AC and CA, and as these are equal and opposite there is equilibrium.



If O be the particle, the lines Om, On, Op parallel and equal to AB, BC, CA respectively, represent the lines of action and the magnitude of the forces.

Again, let the three forces P, Q and R acting on the particle O, represented by OA, OB, OC respectively, be in equilibrium. Complete the parallelogram OADB. Then (Art. 169) OD is the resultant of OA and OB. And since there is equilibrium, OC is equal and opposite to OD (Art. 168). Therefore the sides of the triangle OAD which are respectively parallel to the directions of the forces P, Q, R represent these forces in magnitude. But any other triangle, whose sides are parallel to the triangle OAD, is similar to OAD, and has its sides proportional. Therefore if the sides of any triangle be parallel to the directions of P, Q and R respectively, these sides will be proportional to the forces.



175. Again, since if any one triangle has its sides perpendicular or equally inclined to those of another triangle, the triangles are similar and their sides proportional, it follows that if three forces acting on a particle be in equilibrium, and if a triangle be drawn whose sides are (1) parallel, (2)

perpendicular, or (3) equally inclined to the directions of the forces respectively, the sides of the triangle will be proportional to the forces.

176. If three forces acting on a particle be in equilibrium each force is proportional to the sine of the angle between the directions of the other two.

In the figure of Art. 174 let P, Q, R be the forces, then by trig.

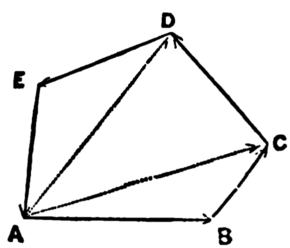
$$\begin{aligned} P : Q : R &:: \sin ODA : \sin DOA : \sin DAO \\ &:: \sin BOC : \sin AOC : \sin BOA \\ &:: \sin \widehat{QR} : \sin \widehat{PR} : \sin \widehat{PQ} \end{aligned}$$

Where $\widehat{QR}$ denotes the angle between the directions of Q and R, $\widehat{PR}$ between P and R, and $\widehat{PQ}$ between P and Q.

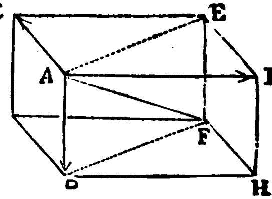
Conversely, if three forces act on a particle, and each force be proportional to the sine of the angle between the other two, and that the direction of each force is outside the angle formed by the directions of the other two, the forces will be in equilibrium.

177. Polygon of Forces.—*If forces which are represented in magnitude and direction by the sides of a polygon taken in order act on a particle they will be in equilibrium.*

Let AB, BC, CD, DE, EA be the forces. Then (Art. 169) the resultant of AB, BC is AC; of AC, CD is AD; and of AD, DE is AE. Therefore all the forces are equivalent to the two EA and AE, and as these are equal and opposite they are in equilibrium.



178. Parallelopiped of Forces.—*If three forces represented in magnitude and direction by three adjacent edges of a parallelopiped act on a particle, their resultant is represented in magnitude and direction by the diagonal of the parallelopiped which is drawn through the intersection of the three sides.*



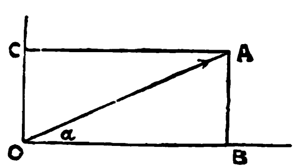
Let AB, AC, AD be the three forces. Then (Art. 169) the resultant of AB, AC is AE; and the resultant of AE, AD is AF. Therefore AF is the resultant of the forces.

AF may be calculated when the numerical values of the three forces are given; for

$$AF = \sqrt{AE^2 + AD^2} = \sqrt{AB^2 + AC^2 + AD^2}.$$

179. *Resolution of Forces.*—The Resolution of Forces is the converse of the Composition of Forces. It is the finding of component forces whose combined effect is equivalent to the given force. Any force may be resolved into two components by taking a line representing the force, and constructing a parallelogram having this line as diagonal. An infinite number of parallelograms can be constructed having the same diagonal, and consequently a force can be resolved in an infinite number of ways. Usually in problems when a force has to be resolved, one of the directions is determined by the problem, and the whole amount of the force resolved in this direction is obtained by taking the direction of the other component at right angles to this direction. Thus in practice a force is resolved by making the line which represents the force the diagonal of a rectangle.

180. The value of the resolved part of a force in any direction can be calculated when the force is given and the angle which its direction makes with the direction in which it is resolved.



Thus if OA be a given force, and a the angle which its direction makes with the horizontal line, then OB is the resolved part of OA in a horizontal direction, and $OB = OA \cos a$. Similarly OC, the resolved part in a vertical direction, $= OA \sin a$.

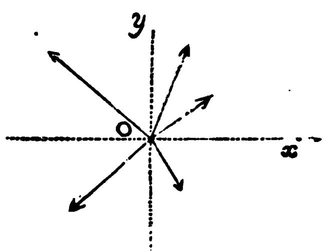
181. *To determine the resultant of any number of forces acting on a particle.*

This may be effected by the following methods (see Art. 32) :—

(*a.*) *By repeated applications of the Parallelogram of Forces.*—Find the resultant of two of the forces by Art. 169, then of this resultant and a third, and so on to the last. The final resultant will be the resultant of all the forces.

(*b.*) *By the Polygon of Forces.*—Construct a figure whose sides taken in order will represent the magnitude and direction of the forces; then the line closing the figure and taken in the opposite direction will represent the resultant. Thus if in figure of Art. 177, AB, BC, CD, DE be lines representing the forces then, as was shown in that Art., AE represents the resultant of these forces.

(*c.*) *By resolving the forces in two directions at right angles to each other and compounding the results.*



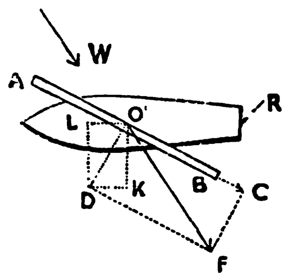
Let O be the particle upon which any number of forces are acting. Through O draw the axes x and y at right angles to each other. Resolve all the forces along x and y. Let the sum of the resolved parts along x be X, and the sum along y be Y, then the resultant of X and Y is the resultant of all the forces. Calling this final resultant R, then by Art. 169—

$$R = \sqrt{X^2 + Y^2}.$$

182. As illustrating the *resolution of forces* we may examine the case of the wind acting obliquely on the sails of a boat, and determine the amount of the wind's force which is available in the direction of the motion of the boat.

From the point O in the centre of the sail draw the line OF representing in magnitude and direction the force of the wind. Resolve OF (Art. 180) into the components OC and OD. The former is parallel to the sail and may be neglected. Resolve

OD into two components, OL and OK, OL represents the amount of the wind's force which is effective in propelling the boat, and OK represents the force which urges the boat sideways. Owing to the shape of the boat the *lee-way* motion produced by the force OK is not great, and it is counteracted by the action of the rudder R.



Examples.

1. Two forces of 20 lbs. and 40 lbs. act on a particle at an angle of 60°. find the resultant.

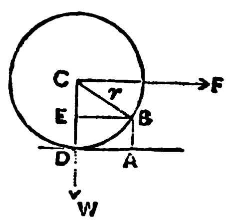
$$\text{Art. 171. } R^2 = 20^2 + 40^2 + 2 \times 20 \times 40 \times \tfrac{1}{2} = 2800$$
$$R = \sqrt{2800} = 20\sqrt{7} \text{ lbs.}$$

This result may also be obtained by Art. 172.

2. The resultant of two equal forces whose directions make an angle of 60° is 8 lbs. What are the forces?

$$\text{Art. 171. } 8^2 = P^2 + P^2 + 2 \cdot P \cdot P \cdot \tfrac{1}{2}$$
$$\therefore 3P^2 = 64 \therefore P = \frac{8}{\sqrt{3}} = \frac{8}{3}\sqrt{3} \text{ lbs.}$$

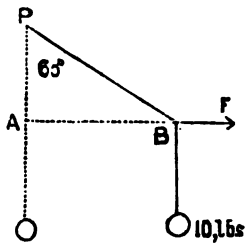
3. A carriage wheel whose weight is W and radius r, moving upon a level road meets with an obstacle of height h. What horizontal force will draw the wheel over the obstacle?



Let F be the force, and let the wheel be on the point of surmounting the obstacle AB. The three forces acting through C are in equilibrium, namely the force F, the weight of the wheel W, and the reaction of the point B. Therefore, Art. 174, F : W : : EB : CE. But $CE = r - h$ and $EB = \sqrt{CB^2 - CE^2} = \sqrt{2rh - h^2} \therefore F : W : : \sqrt{2rh - h^2} : r - h$ and

$$F = W\frac{\sqrt{2rh - h^2}}{r - h}$$

4. A weight of 10 lbs. is suspended from a fixed point by a flexible string; what horizontal force applied to the string will draw the upper portion aside through an angle of 60°

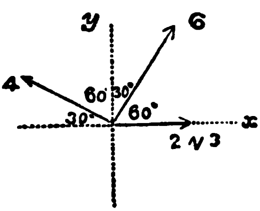


Let F= the force. The point B is at rest under the three forces F, the tension of the string, and the weight of 10 lbs. Therefore, Art 174—

$F : 10 : : AB : AP$

$: : \sqrt{3} : 1 \therefore F = 10\sqrt{3} = 17{\cdot}32$ lbs.

5. Three forces of $2\sqrt{3}$ lbs., 6 lbs., and 4 lbs. act on a particle. The angle between the first and second is 60°, and the angle between the second and third is 90°: find the resultant.



Resolving horizontally and vertically (Art. 181)—

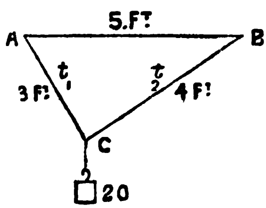
$$X = 2\sqrt{3} + 3 - 2\sqrt{3} = 3$$

$$Y = 3\sqrt{3} + 2$$

$$\therefore R = \sqrt{X^2 + Y^2} = \sqrt{60{\cdot}784} = 7{\cdot}8.$$

6. The ends of two cords AC and BC, which are respectively 3 ft. and 4 ft. long, are joined together at C, and their other ends fastened to A and B in the same horizontal line.

A weight of 20 lbs. is suspended from C, and ACB is a right angle: find the tension in each cord.



Let t_1 t_2 be the tensions. The sides of the triangle ACB are respectively perpendicular to the three forces acting on the point C. Therefore (Art. 175)—

$t_1 : 20 : : BC : AB : : 4 : 5 \therefore t_1 = 16$ lbs.

$t_2 : 20 : : AC : AB : : 3 : 5 \therefore t_2 = 12$ lbs.

Exercises.

1. Two forces which are as $3 : 4$ act on a particle in directions at right angles to each other, and have a resultant of 15 lbs.: what are the forces?

2. Can three forces of 12, 6 and 4 lbs. keep a particle at rest?

3. If forces of 12, 8, and 4 lbs. keep a particle at rest, how are they acting?

4. A uniform rod whose weight is W is suspended by two equal strings attached to its ends and to a fixed point, the rod and strings forming an isosceles right angled triangle. Compare the tension of each string with the weight of the rod.

5. If in the foregoing case the strings and rod formed an equilateral triangle, compare the tension with W.

6. Two forces act at right angles to each other on a particle. The resultant is 25 lbs., and one of the components is 15 lbs., what is the other?

7. The resultant of two components of 21 lbs. and 28 lbs. is 35 lbs.: what is the angle between the components?

8. A cord is passed through a ring fixed in a wall, and the ends are pulled each with a force of 20 lbs.; if the angle between the two parts of the cord be $120°$, what is the pressure on the ring?

9. A weight is moved along the ground by a cord which is pulled with a force of 30 lbs. at an angle of $30°$ to the horizon: what force acting horizontally would produce the same effect?

10. Forces of 12 lbs. and 4 lbs. act along two adjacent sides of a square: what is their combined effect on a particle at the angular point?

11. Three forces of 20, 30, and $12\sqrt{7}$ lbs. act on a particle. The angle between the directions of any two is $120°$: what is their resultant?

12. Forces of 10 lbs. and 12 lbs. act at an angle whose cosine is $\frac{1}{8}$: find the resultant?

13. Two forces of 2 lbs. and 3 lbs. act on a particle at an angle of $45°$: find the resultant.

14. If the directions of two forces be inclined to each other at an angle of $135°$, find the ratio between them when the resultant is equal to the less force.

15. A weight of 20 lbs. hangs by a cord from a fixed point. A horizontal force applied to a point in the cord pulls it aside till the upper portion forms an angle of $45°$ with its former direction: find the force.

16. In the foregoing case find the pull on the fixed point.

17. Two weights of 5 lbs. each are attached to the ends of a string which is passed over two pulleys in a horizontal line, and another weight of $5\sqrt{3}$ lbs. hangs from a ring which slides freely on the cord. When there is equilibrium, find the angle between the portions of the string on each side of the ring.

18. An elastic ring is stretched round three pegs which are driven into a board at the angular points of an equilateral triangle: what is the pressure on each peg if the tension of the elastic band be 10 lbs.?

19. Can three forces of 6, 3, and 10 lbs. acting on a particle be in equilibrium?

20. Three adjacent sides of a regular hexagon taken in order represent the magnitude and direction of the forces acting on a particle: what is the resultant?

21. Cords 6 ft. and 8 ft. long are fastened at their upper ends to two

hooks in a horizontal line, the lower ends being knotted together. A weight of 20 lbs. is suspended from the knot, and the cords are at right angles to each other: what is the tension in each cord?

22. Two forces act on a particle in directions which make an angle whose cosine is $\frac{1}{10}$: find the resultant, the forces being 5 lbs. and 4 lbs. respectively.

23. In the arrangement of Art. 173, if the weights P and Q be 10 and 12 ozs. respectively, and the angle AOB 120°, what must be the weight of R so that there may be equilibrium.

24. If in the same system the weights P and Q are 12 and 16 ozs., and the angle AOB 90°, what is the value of R when there is equilibrium?

25. Three forces of 4, 6, and 8 lbs. act on a particle. The angle between the first and second is 60°, and between the second and third is 90°: what is the resultant?

26. Three forces act perpendicularly to the sides of a triangle at the middle points, and each is proportional to the side on which it acts. Show that they will equilibrate each other.

27. Three forces, the magnitude, and directions of which are given act on a particle: show how their resultant may be found.

28. A force of 100 lbs. acts in a direction inclined to the horizon at an angle of 30°: what is the effect of this force in the horizontal direction, and what in the vertical?

29. Three forces of 99, 100, and 101 lbs. respectively, act upon a particle in directions making an angle of 120° with each other successively: find the magnitude of the resultant, and the angle which its direction makes with the force of 100 lbs.

30. Forces of 2 lbs. and $3\sqrt{3}$ lbs. act on a particle at an angle of 30°: find the resultant.

31. Forces of 17 and $24\sqrt{2}$ lbs. act at an angle of 135°: find the resultant.

32. Two forces, one of which is 4 lbs., act on a particle at an angle of 30°, and produce a resultant of 14 lbs.: find the other component.

33. Three smooth pegs, A, B, C, are arranged in a vertical wall at the angular points of an equilateral triangle whose base BC is horizontal. A string is passed round the pegs and a weight of 10 lbs. is attached to each end of the string: find the pressures on the pegs.

34. If in the preceding case the angle BAC be a right angle and the triangle isosceles, find the pressures on the pegs.

35. Two men pull each with a force of 80 lbs. in directions inclined to each other at an angle of 60° at the ends of a rope passing round a smooth peg: find the pressure on the peg.

36. Forces of 3, 4, 5, and 6 lbs. act on a particle at the centre of a square in directions tending to the angular points respectively: find the resultant force.

37. Two cords, AO and BO, are knotted together at the point O, and have their other ends fastened respectively to two points A and B in the same horizontal line, a weight of 40 lbs. being suspended from O. The

cord AO is 12 inches long, BO 16 inches, and the angle AOB is a right angle: find the tensions in the cord.

38. The two diagonals of a parallelogram represent the forces acting on a particle: what is the resultant.

39. Three forces of 4, 5, and 6 lbs. act on a particle. The direction of each is at right angles to the directions of the other two: find the resultant.

40. Forces of 10, 10, 8, 6, and $4\sqrt{3}$ lbs. act on a particle. The angles between the directions of the forces taken in order are 45°, 75°, 60°, and 90°: find the resultant of all the forces.

41. A weight is drawn along the ground by two ropes which are in the same vertical plane. The ropes make angles of 45° and 60° with the horizon, and are pulled by forces of 20 lbs. and 30 lbs. respectively: what single horizontal force is equivalent to the two tensions?

42. When a horse is employed to tow a boat along a canal, the towing rope is usually of considerable length: why is a long rope better than a shorter one?

43. A weight of 24 lbs. is suspended by two flexible strings, one of which is horizontal and the other inclined at an angle of 30° to the vertical. What is the tension in each string?

CHAPTER XIII.

PARALLEL FORCES.

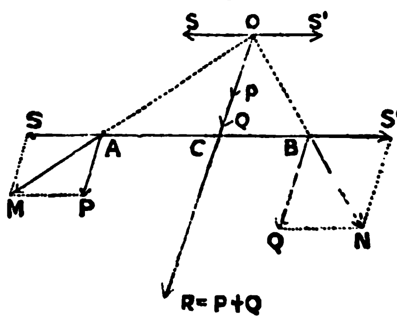
183. Forces acting on a particle, or those whose lines of action meet at the same point of a rigid body, have been treated of in the last chapter. We have now to consider forces acting on different points of a rigid body, and whose directions are *parallel.*

Parallel forces are called *like* when they act in the same direction, and *unlike* when they act in opposite directions.

184. Experiment has shown that the point of application of a force may be transferred to any other point in the line of action of the force without altering its effect. This has been called the *transmissibility of force.*

185. *Composition of like Parallel Forces.*—The magnitude, direction, and point of application of the resultant of two like parallel forces are determined by the following proposition:—

The resultant of two like parallel forces is equal to their sum, parallel to their direction, and applied at a point found by dividing the line between them inversely as the forces.



Let A and B be two points of a rigid body (the body is not shown in the figure) at which the parallel forces P and Q act respectively. Apply two equal and opposite forces S and S′, one acting at A, the

other at B. Since these forces are equal and opposite, they equilibrate each other, and the effect of the forces P and Q is unaltered. Find (Art. 169) AM the resultant of P and S, and BN the resultant of Q and S′. The forces AM and BN may be transferred (Art. 184) to the point O. Resolve AM at O into its two components parallel to their original directions. We thus obtain a force S acting at O, and a force P acting along OC parallel to AP. Similarly resolve BN into its components, and we obtain S′ and Q. S and S′ being equal and opposite equilibrate each other. There remain P and Q acting along OC. These are equivalent (Art. 167) to a resultant P + Q which passes through a point C in AB. The position of C may be determined as follows :—

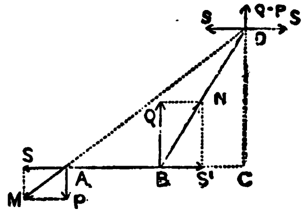
The triangle OCA has its sides parallel respectively to the forces P, S, and AM ; and the triangle OCB has its sides respectively parallel to Q, S′, BN. Therefore (Art. 174) $\frac{P}{S}=\frac{OC}{AC}$ and $\frac{Q}{S'}=\frac{OC}{BC}$. Dividing the first equation by the second, $\frac{P}{Q}=\frac{BC}{AC}$, or P : Q :: BC : AC. Hence the point C divides AB inversely as the forces.

186. Composition of *unlike* Parallel Forces.—*The resultant of two unlike parallel forces is equal to their difference, parallel to their line of action, acts in the direction of the greater force, and is applied at a point found by producing the line between them through the point of application of the greater force, so that the whole produced line is to the produced part inversely as the forces.*

Let the forces be P acting at A and Q at B. Apply two equal and opposite forces S and S′. Compound as in Art. 185, and let the resultants act at O. Resolve into their components. S and S′ equilibrate, and there remain Q and P acting in opposite directions in the line OC. Their resultant is therefore Q − P, acting in the direction of Q. Let the resultant act at C. The position of C

may be found as follows:—
The sides of the triangle OCA are parallel to P, S, and AM, and those of OCB to Q, S′, and BN. Therefore, as in last Art.,

$\frac{P}{Q} = \frac{BC}{AC}$, or Q : P : : AC : BC.



187. The two foregoing propositions may be experimentally verified as follows:—

Fig. 1.

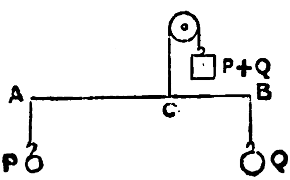
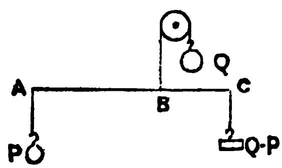


Fig. 2.



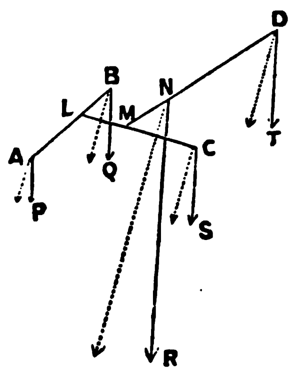
Two weights P and Q are attached (Fig 1.) to the extremities of a light rod, whose weight may be neglected. These weights are parallel forces acting vertically downwards, and are equivalent to a resultant acting at some point of the bar, which would be equilibrated by an equal and opposite force acting vertically upwards at the same point. A string is passed over a pulley and is attached by one end to a ring which slides on the rod, and by the other to a weight P + Q; and the ring is moved until AB remains horizontal when suspended by the string. Then the tension of the string, which is P + Q, is equal to the resultant of the weights P and Q; and the point C is found to divide AB, so that AC : CB : : Q : P.

Again in Fig. 2, a weight P acts downwards at A, and a weight Q upwards at B, and in order to maintain equilibrium

a weight Q − P must be attached at a point C in the bar, so that AC : CB : : Q : P.

188. *Couples.*—The point of application of the resultant of two unlike parallel forces is found by Art. 186. If the two unlike parallel forces be *equal*, the determination of this point becomes impossible; for in this case we should have to produce the line between the forces so that the whole line produced would be equal to the produced part. Two equal unlike parallel forces have therefore no resultant. They form what is called a *Couple*. A couple tends to produce rotation in the body on which it acts; and it cannot be balanced by any single force.

189. *To find the magnitude, direction, and point of application of the resultant of any number of parallel forces which are either in the same, or in different planes.*



Let the forces be P, Q, S, T, acting at the points A, B, C, D, respectively. Join AB and divide it in L inversely as the forces P and Q. Then (Art. 185) the resultant of P and Q is P + Q acting at L parallel to the forces. Join LC and divide it at M inversely as the forces P + Q at L, and S at C. Then the resultant of P, Q, and S is a force P + Q + S acting parallel to the forces at M. Join MD and divide it at N inversely as P + Q + S at M, and T at D. Then the resultant R = P + Q + S + T acts parallel to the forces at the point N. In the same way we may determine the resultant and its point of application of any number of parallel forces.

190. *Centre of Parallel forces.*—It will be seen that in the foregoing demonstration the determination of the point N is independent of the direction of the parallel forces, so that

if the whole system of forces be turned into the direction indicated by the dotted lines, the position of N would be unchanged. In every system of parallel forces there is, therefore, a point through which the direction of the resultant always passes, no matter what may be the position of the system, and this point is called the Centre of the System of Parallel Forces.

191. In Art. 189 the forces were assumed to be like. If some of the forces act in a direction opposite to the others we have to employ the principles of Art. 186 in addition to those of Art. 185. For instance, in the preceding demonstration if the force S be unlike to P and Q, then the point M is found by Art. 186, by producing CL to M, so that the whole line produced is to the produced part as $P + Q : S$. Or the forces may be arranged into two groups, each consisting of like forces, and the resultant of each determined as in the preceding demonstration, and finally the resultant of these two by Art. 186.

192. By proceeding in this manner we shall obtain with every system of parallel forces some one of the following results :—(1) A single resultant acting at a determinate centre; or (2) two equal and opposite forces forming a couple; or (3) the resultant vanishes, the forces being in equilibrium.

193. The centre of a system of parallel forces may always be found by the geometrical method of Art. 189. If, however, the position of the centre has to be calculated, this method is laborious. Another mode better adapted for calculation will be explained further on.

194. *Resolution of Parallel Forces.*—By means of the principles demonstrated in Arts. 185 and 186, a force can be resolved into two or more component forces acting at assigned points. For example, if a force R be given, acting at the point C, it can be resolved (Art. 185) into two parallel forces acting at A and B respectively, by dividing R into two forces inversely proportional to AC and CB. And (Art. 186) a force $Q - P$ acting at C can be replaced by two forces Q at B and P at A such that $Q : P :: AC : BC$.

Examples.

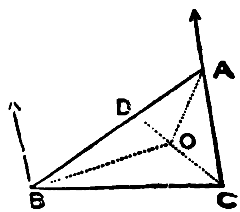
1. Unlike parallel forces of 15 lbs. and 5 lbs. act at points 6 inches apart: find their resultant, and its point of application. By Art. 186 the resultant $= 15 - 5 = 10$ lbs.

Let x = distance of point of application of resultant from the greater force; then, Art. 186,

$$6 + x : x :: 15 : 5$$
$$15x = 30 + 5x \therefore x = 3.$$

The point of application of the resultant is 3 inches from that of the 15 lbs.

2. ABC is a triangle, O a point in it, and like parallel forces P and Q act at A and B of such a magnitude that P : Q :: triangle BOC : triangle AOC. Show that if CO be produced to intersect AB in D, the resultant acts at D.



$$P : Q :: BOC : AOC \text{ by hypothesis.}$$
$$\left.\begin{array}{l} :: BOD : AOD \\ :: BD : DA \end{array}\right\} \text{ by Euc. VI., 1.}$$

$\therefore$ Art. 185, D is the point of application of the resultant.

Exercises.

1. Like parallel forces 10 and 6 act at two points 4 feet apart: find their resultant and its point of application.

2. Weights of 5 lbs. and 7 lbs. are placed at the extremities of a rod 36 inches long: upon what point will the rod balance, and what will be the pressure upon the point?

3. Two unlike parallel forces 12 and 16 lbs. act at two points 1 foot apart: find their resultant and its point of application.

4. Two unlike parallel forces 7 and 10 lbs. act at points 6 inches apart: find resultant and point of application.

5. Like parallel forces of 1, 2, 3, and 6 lbs. act respectively at the corners A, B, C, D of a square, whose side is 10 feet: find the magnitude and point of application of the resultant.

6. A cask whose weight is 224 lbs. is slung from a pole, the ends of which rest on the shoulders of two men who carry the weight between them. The cask is double as far from the first as from the other: what weight does each support?

7. ABC is an isosceles triangle having a right angle at C, and CD is the

perpendicular on the hypotenuse AB. Three equal forces act along the sides and in the directions AB, BC, CA. Show that their resultant acts at a point E found by producing DC, so that DE:CE: : $\sqrt{2}$: 1.

8. ABC is a triangle and O any point within it. Like parallel forces proportional respectively to the triangles BOC, COA, and AOB act at A, B, and C respectively: show that O is the point of application of the resultant of these forces.

9. Resolve a force of 8 lbs. into two like parallel forces of 8 inches apart, and one of them 3 inches from the given force.

10. The resultant of two unlike parallel forces which are 6 inches apart is 3 lbs., and it acts at a distance of 8 inches from the greater force: what are the forces?

CHAPTER XIV.

MOMENTS OF FORCES.

195. If a point in a body be fixed, and a force act on the body, it will tend to turn it round the fixed point, unless the direction of the force pass through the point.

Fig. 1.

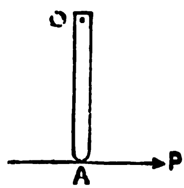
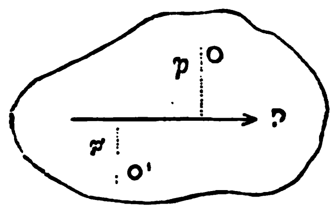


Fig. 2.



Thus if OA (Fig. 1) be a rod capable of turning round a fixed pivot at O, and a force P act at the other extremity, it will cause the rod to rotate round O. Similarly, if the body in Fig. 2 be acted on by a force P, and a point O in the body be fixed, the force will tend to turn it round O. If, instead, the point O′ were fixed, the force would tend to produce rotation round O′. But if a point in the line of action of P be fixed, the force will be met by the reaction of the fixed point, and no rotation will be produced.

Thus whenever a force acts on a body, and some point not in the line of action of the force be fixed, the force tends to turn the body round the point.

196. Experiment shows that the effect of the force in producing rotation varies with its magnitude and with the length of the perpendicular from the fixed point upon the

line of action of the force. Thus the effect of any force in tending to produce rotation round a fixed point is measured by the product of the force into the perpendicular distance of its line of action from the point. This product is called the *Moment* of the force round the point.

Thus in Fig. 1, Art. 195, the moment of P with respect to the point O is the power of P to produce rotation round O, and is measured by the product of the number of units of magnitude in P into the number of units of length in OA. This is the *algebraic measure* of the moment of a force round a point. Hence if in Fig. 2 the perpendicular from O on the direction of P be p, the moment of P round O is Pp. Similarly the moment of P round O′ is Pp'.

A *geometrical measure* of the moment of a force may also be found. If the force in Fig. 2 Art. 195 be represented by the line P, and if the extremities of this line be joined with O, a triangle is formed the area of which is $\frac{1}{2}$Pp. Therefore the moment of a force round a point is geometrically measured by double the area of a triangle whose vertex is the point and whose base is the line representing the force.

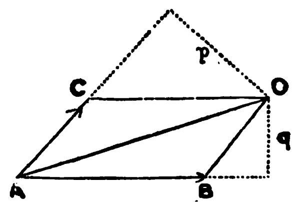
197. Rotation can take place in two directions. In Fig. 2. Art. 195, if O be the fixed point, the rotation is opposite to that of the hands of a clock, or left-handed; while if O′ be fixed, the rotation is in the direction of the hands of a clock, or right-handed. Moments are therefore of two distinct kinds, and it is convenient to call one of these positive and the other negative. It is indifferent which we call positive; so that in any problem we can fix upon moments in one direction as positive, and in the other as negative, but throughout the problem we must maintain this distinction.

Hence, since moments will be either + or −, the sum of any number of moments will always mean their algebraic sum. The sum of two equal moments, one of which is + and the other −, is zero.

198. *Moments of Forces acting on a Particle.*

Prop. 1.—*If the directions of two forces meet at a point,*

their moments with respect to any point situated on the line of action of their resultant are equal and unlike. And conversely, *if the moments of two such forces with respect to any point be equal and unlike, the point is situated on the line of action of the resultant.*

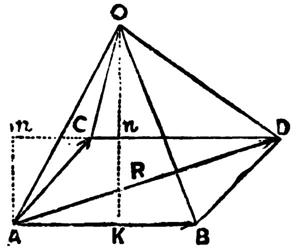


Let the directions of two forces P and Q meet at A, and let O be any point in the line of action of the resultant of P and Q, then the moments of P and Q round O are equal and unlike.

Since O is a point in the direction of the resultant, any portion of the line passing through A and O may be taken for the resultant, and the adjacent sides of the parallelogram constructed on this line as diagonal will represent the forces P and Q (Art. 169). Take AO to represent the resultant, and through O draw OC and OB parallel to AB, AC respectively; then AC is equal to P and AB to Q. But the triangle AOC is equal to AOB, and (Art. 196) AOC $= \frac{1}{2}$ moment of P round O, and AOB $= \frac{1}{2}$ moment of Q round O; therefore the moments of P and Q round O are equal, and they are unlike since they tend to produce rotation in opposite directions. Hence if p and q be the respective perpendiculars from O on the directions of P and Q, $Pp = Qq$.

Again if Pp and Qq are equal and unlike, O is a point on the line of action of the resultant: The same construction being made, the triangle AOC = AOB (Euc. I., 34) $\therefore \frac{1}{2}AC \cdot p = \frac{1}{2}AB \cdot q$, therefore $AC \cdot p = AB \cdot q$. But by hypothesis $Pp = Qq$; therefore AC : AB :: P : Q. Hence AO is the resultant of P and Q, and O is therefore on the resultant.

199. Prop. 2. *If the directions of two forces meet in a point, the algebraic sum of the moments of the forces round any point in their plane is equal to the moment of their resultant round the same point.*



Let P and Q be two forces and R their resultant represented respectively by the lines AC, AB, AD, and let the point O fall without the angle CAB. Then the moments round O are all like, and the moment of R round O is equal to the arithmetical sum of the moments of P and Q round O. This is established by showing that the triangle OAD = the sum of the triangles OAC, OAB, these triangles being respectively the halves of the moments round O (Art. 196). The triangle OAD = OAC + OCD + ACD. Draw the perpendicular Ok meeting CD at n, and the perpendicular Am meeting DC produced. The triangle OCD = $\frac{1}{2}$CD · On (Euc. I., 41). The triangle ACD = $\frac{1}{2}$CD · Am = $\frac{1}{2}$CD · nk. Therefore OCD + ACD = $\frac{1}{2}$CD (On + nk) = $\frac{1}{2}$CD · Ok. But OAB = $\frac{1}{2}$AB · Ok $\therefore$ OAB = OCD + ACD $\therefore$ OAD = OAC + OAB. Hence, doubling both sides of the equation, the moment of R round O = sum of moments of P and Q round O.

Again, let O fall within the angle CAB, then the moments of P and Q are *unlike*, and the moment of R round O is equal to the arithmetical difference of the moments of P and Q round O. For it may be shown in a similar manner to the foregoing demonstration that the triangle OAD is equal to the difference of the triangles OAC and OAB. Hence the moment of the resultant round any point is equal to the algebraic sum of the moments of the forces round the same point.

The proposition of Art. 198 is only a case of this proposition. When the point O falls on the line of action of the resultant, the moment of the resultant round O is zero, and, therefore, the algebraic sum of the moments of P and Q round O is zero. Hence the moments of P and Q are equal and of opposite sign.

200. Prop. 3. *If any number of forces in the same plane act on a particle, the algebraic sum of the moments round any*

point in the plane is equal to the moment of the resultant round the same point.

By the last Art. the proposition holds for two of the forces. Then, taking the resultant of these two and a third force, it may be shown in the same way that the proposition holds for *three* forces: and so on for any number of forces.

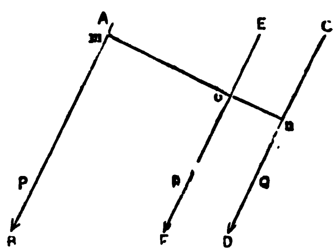
201. Prop. 4. *If any number of forces acting on a particle be in equilibrium, the algebraic sum of their moments with respect to any point in the plane is equal to zero.*

By finding the resultant of two of the forces, then of that resultant and a third, and so on, we should finally obtain two forces equal and opposite to each other. The moments of these with respect to any point would be equal and opposite, and hence the sum of the moments would be zero.

Or, since the forces are in equilibrium, their resultant is zero, and therefore the sum of the moments of the forces round any point, being equal to the moment of the resultant round the point, must also be zero.

202. *Moments of Parallel Forces.*

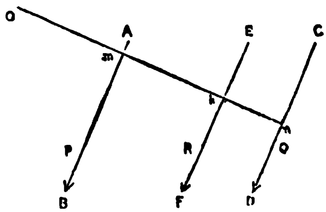
Prop. 1. *The moments of two like parallel forces round any point in the direction of their resultant are equal and unlike.* And, conversely, *if the moments of two like parallel forces with respect to a point be equal and unlike, the point is on the line of action of the resultant.*



Let forces P and Q act along the parallel lines AB and CD, and let EF be the line of action of the resultant. Then if O be a point on this line, the moments of P and Q round O are equal and unlike. Through O draw mOn at right angles to the parallel lines. Then (Art. 185), $P : Q :: On : Om$ ∴ $P \cdot Om = Q \cdot On$; and these moments are unlike.

Again let $P \cdot Om = Q \cdot On$. Then $P : Q :: On : Om$, and (Art. 185) the point O is on the direction of the resultant.

203. Prop. 2. *The algebraic sum of the moments of any two parallel forces round any point in their plane is equal to the moment of their resultant round the same point.*



First let the forces be *like*, and let the point lie without the lines of action of the forces. Let the forces P, Q, and resultant R act along the lines AB, CD, EF respectively, and let O be the point. From O draw Omkn perpendicular to the parallel lines. Then $P \cdot Om + Q \cdot On = P\ (Ok - mk) + Q\ (Ok + nk) = P \cdot Ok - P \cdot mk + Q \cdot Ok + Q \cdot nk = (P + Q)\ Ok - P \cdot mk + Q \cdot nk = (P + Q)\ Ok = R \cdot Ok$. Since, (Art. 185) $P \cdot mk = Q \cdot nk$.

In the same way it may be shown that the moment of R round O is equal to the arithmetical difference of the moments of P and Q, when the point O is taken between the lines of action of P and Q.

And similarly it may be shown that the proposition is true for unlike forces wherever the point O is taken.

204. Prop. 3. *The algebraic sum of the moments of any number of parallel forces round any point in the plane is equal to the moment of the resultant round the same point.*

The proof is similar to that of Art. 200.

205. Prop. 4. *If any number of parallel forces be in equilibrium the algebraic sum of their moments round any point in the plane is equal to zero.*

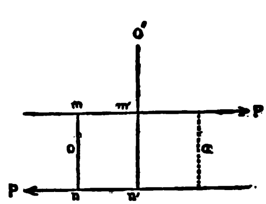
The proof is similar to that of Art. 201.

206. *Moments of Couples.* (See Art. 188.)

A *couple* consists of two equal unlike parallel forces. The *arm* of the couple is the perpendicular distance between the directions of its forces. The *moment* of the couple is the product of one of the forces into the arm. The *axis* of the couple is a straight line perpendicular to the plane of the couple and having a length proportional to its moment. Couples are *like* when they tend to turn the body in the

same direction, and *unlike* when they tend to turn it in opposite directions.

207. Prop. 1.—*The algebraic sum of the moments of the two forces which form a couple is constant round every point in their plane, and is always equal to the moment of the couple.*



Let PP be the couple, and a its arm. Take any point O between the lines of action of the forces; then the sum of the moments $P \cdot Om$ and $P \cdot On = P \cdot mn = P \cdot a =$ moment of the couple.

Again, take any point O′ outside the lines of action. Then the algebraic sum $= P \cdot O'n' - P \cdot O'm' = P \cdot m'n' = P \cdot a =$ moment of couple.

208. Prop. 2.—*Two unlike couples in the same plane whose moments are equal will balance each other.*

This proposition may be regarded as a corollary from the definition of the moment of a couple. Each couple will have the same effect in turning the body, and as the couples are equal and opposite they will balance.

209. From the last Art. it may be inferred that like couples of equal moments in the same plane will have equal effects, and will be equivalent to one like couple having a moment double that of either of the couples.

That if there be any number of like couples in the plane they are equivalent to a like couple having a moment equal to the sum of the moments of the couples.

And that if there be any number of couples positive and negative in one plane, the algebraic sum of the moments of the couples is equal to the moment of a couple whose effect is equal to the combined effect of all the couples.

210. *Forces in one plane acting on a particle.*

A system of forces acting in one plane on a particle will either have a single resultant, or will be in equilibrium.

For finding the resultant of two of the forces, then of that

resultant and a third force, and so on, we shall finally arrive at either of the following results :—(1) A single resultant; or (2) One of the forces of the system equal and opposite to the resultant of all the others, so that the forces are in equilibrium.

211. *Conditions which hold with a system of forces in one plane acting on a particle and in equilibrium :—*

(1.) *The sum of the forces resolved in any direction vanishes.*

(2.) *The sum of the moments of the forces round any point in the plane vanishes.*

(1) follows from last Art., and (2) from Art. 201.

212. If two forces be in equilibrium, it is evident they must act in the same straight line and in opposite directions.

If three forces which are not parallel are in equilibrium, their directions must meet in a point. For let the directions of two of them meet at a point, the resultant of these two passes through the point, and as this resultant is equal and opposite to the remaining force its direction also passes through the point.

213. *Forces in one plane on a rigid body.*

A system of forces in one plane acting on a rigid body must be (1) *equivalent to a single resultant; or* (2) *equivalent to a couple; or* (3) *be in equilibrium.*

For, compounding two of the forces, then that resultant and a third force, and so on, we shall finally obtain some one of the following results :—(1) A single resultant; (2) A couple; (3) One of the forces equal and opposite to the resultant of all the others. In (1) the sum of the moments of the forces round any point in the plane is equal to the moment of the resultant; and this sum vanishes only for points on the line of the resultant. In (2) the sum of the moments of the forces round any point in the plane never vanishes, and is always the same for every point in the plane. In (3) the sum of the moments round any point in the plane always vanishes.

214. *Conditions which hold with a system of forces in equilibrium acting in one plane :—*

(1.) *The algebraic sum of the forces resolved in any direction vanishes.*

(2.) *The algebraic sum of the moments of the forces round any point in the plane vanishes.*

These conditions follow from last Art.

215. *Conditions which determine the equilibrium of a system of forces in one plane.*

A system of forces acting on a rigid body in one plane will be in equilibrium :—

(1.) *If the sum of the resolved parts of the forces in any two directions vanishes, and the sum of the moments of the forces round any point in the plane also vanishes.*

(2.) *If the sum of the moments round two points in the plane vanishes, and the sum of the forces resolved in the direction of the line joining these two points vanishes.*

(3.) *If the sum of the moments of the forces round three points in the plane which are not in a straight line vanishes.*

The conditions laid down in each of the foregoing cases exclude the hypothesis that the forces have a single resultant, or that they are equivalent to a couple, and therefore they must be in equilibrium (Art. 213). If the forces had a single resultant, the sum of their moments would vanish only round points in the line of the resultant; and the sum of the resolved forces would vanish only when the forces are resolved at right angles to the line of the resultant. Now in (1) the sum of the resolved parts in *two* directions vanishes; in (2) if there were a resultant it would be in the line joining the two points, and by hypothesis the sum of the resolved forces in this direction vanishes; in (3) the three points cannot all lie in the line of a resultant. Hence in none of the cases can the forces have a single resultant.

Again the forces cannot be equivalent to a couple in any of the cases, for the sum of their moments vanishes round a point in the plane.

The forces are therefore in equilibrium.

216. We proceed to apply the principles which have been explained to solve some simple problems dealing with forces in one plane. The bodies on which the forces act are supposed to be rigid and perfectly *smooth*. All reactions between such bodies are at right angles to the surfaces in contact. The reactions of *rough* surfaces will be considered in a future chapter.

When we have to consider problems involving several unknown forces, we may always eliminate any one of the forces which we do not require by taking moments round some point in its direction, or by resolving the forces at right angles to its line of action. And when the directions of two or more forces intersect at a point, by taking moments round this point we obtain an equation not containing any of these forces.

EXAMPLES.

1. Three like parallel forces act on a bar 5 feet long; one of 5 lbs. at one end, 10 lbs. at the middle point, and 9 lbs. at the other end: find the magnitude and point of application of the resultant.

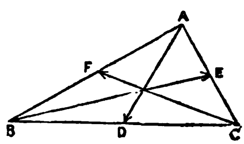
$$\text{Resultant} = 5 + 10 + 9 = 24 \text{ lbs.}$$

Let x = distance of the point of application of the resultant from the end where the 9 lbs. force acts. Taking moments round this point (Art. 204),

$$24x = 5 \times 5 + 10 \times 2\tfrac{1}{2} \therefore x = 2\tfrac{1}{12} \text{ feet.}$$

The resultant is 24 lbs. and acts at a distance of $2\frac{1}{12}$ ft. from one end.

2. Forces are represented in magnitude and direction by the lines drawn from the angular points of a triangle to the points of bisection of the opposite sides. Show that they are in equilibrium.



Take moments round A. The moment of AD = 0; the moment of BE = − twice triangle BAE = − BAC; the moment of CF = + twice CFA = + BAC. Therefore the sum of the moments vanishes round A. Similarly it may be shown that the sum of the moments vanishes round B, and vanishes round C.

Therefore (Art. 215) the forces are in equilibrium.

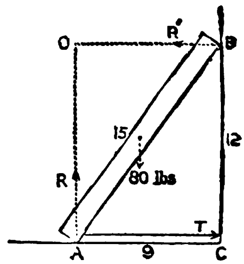
3. A uniform beam 15 feet long and weighing 80 lbs. rests with one end against a smooth vertical wall, and the other end upon a smooth horizontal floor, the latter end being connected to the base of the wall by a string 9 feet long.

Find the tension of the string, and the reactions at the ends of the beam.

Let AB be the beam, T the tension of the string, and R and R′ the reactions at A and B respectively. Resolve vertically and horizontally and take moments round A; (Art. 216),

$$R - 80 = 0 \quad . \quad . \quad . \quad . \quad (1)$$
$$T - R' = 0 \quad . \quad . \quad . \quad . \quad (2)$$
$$R' \times OA = 80 \times \tfrac{1}{2} AC \quad . \quad (3)$$

From (1) $R = 80$ lbs.; from (2) $T = R'$; from (3) $R' = 30$ lbs.



Exercises.

1. If three forces act at a point, and produce equilibrium, they must lie in the same plane.

2. When will two couples produce equilibrium?

3. Three forces in one plane are in equilibrium: show that either they are parallel forces, or that their directions all pass through the same point.

4. Five equal parallel forces act at five of the angles of a regular hexagon whose side is 10 inches: find the point of application of their resultant.

5. Four parallel forces 1, 2, 4, and 7 lbs. act on a bar at distances of 3, 5, 7, and 9 feet respectively from one end: find the magnitude of the resultant and its distance from the end of the bar.

6. Four parallel forces of 3, 4, 5, and 7 lbs., act on a bar at distances of 2, 4, 6, and 8 feet respectively from one end, find the magnitude of the resultant and its distance from the end of the bar.

7. Parallel forces of 2 lbs., 6 lbs., and 4 lbs. act respectively at one end, the middle, and the other end of a rod 1 foot long: what is their resultant, and how far is its point of application from the first end of the rod?

8. Forces acting in one plane are represented in magnitude and position by the sides of a polygon taken in order: show that they are equivalent to a couple.

9. A uniform beam AB whose length is l and weight W rests with its end A on a smooth horizontal plane AC, and its end B on a smooth plane CB inclined to AC at an angle of 120°, CA being equal to CB. A string fastened to the end A and to C keeps the beam from sliding: find its tension.

10. A bundle is attached to one end of a stick which rests on a man's shoulder, the other end being held by his hand. Show that as he diminishes the length between his hand and shoulder the pressure on his shoulder increases while the pressure on the ground remains unaltered.

11. Two pegs are placed nearly in a horizontal line, and a uniform rod lies horizontally between them resting upon one, and having one extremity

just underneath the other. The rod is 12 feet long, and the pegs are 4 feet asunder: compare the pressures on them.

12. A uniform rod 24 inches long and weighing 20 ozs. is placed upon two pegs arranged horizontally at a distance of 12 inches, so that the middle point of the rod is equidistant from each peg. Where must a weight of 24 ozs. be placed so that the pressure on one of the pegs may be 8 ozs.?

13. A uniform beam AB whose weight is 100 lbs. has its two extremities A and B resting on two smooth planes AC and CB, AC being horizontal, and CB inclined to AC at an angle of 120°. The extremity A is tied with a string to a point C in the intersection of the planes, and the distances AC, CB are equal: find the tension in the string, and the pressure on each plane.

14. Two planes AC and CB are inclined at an angle of 120°, AC being horizontal, and a uniform beam AB whose weight is 32 lbs. rests between them making an angle of 30° with each plane, and having the lower end A connected with the point C by a string: find the tension of the string and the pressure on each plane.

15. A bent lever whose weight is neglected consists of two arms CA and CB inclined to each other at an angle of 120°. AC is 3 inches long, CB 12 inches, and a weight of 20 ozs. is suspended at A: what weight must be attached to the extremity B, in order that the lever may rest with the arm AC horizontal?

16. A uniform beam AB whose weight is 50 lbs. is suspended by two cords AC and BC attached to its extremities, and to a fixed point C. The cord BC is vertical and equal in length to AB: required the tensions in the cords.

17. Solve Exam. 3 to Chap. XII. by taking moments round the point B.

18. A pole 12 feet long weighing 25 lbs. rests with one end against the foot of a wall, and from a point 2 feet from the other end a cord runs horizontally to a point in the wall 8 feet from the ground: find the tension of the cord and the pressure of the lower end of the pole.

CHAPTER XV.

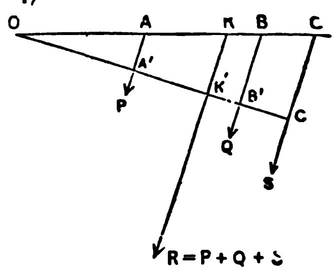
CENTRE OF PARALLEL FORCES.

217. The centre of a system of parallel forces has been defined in Art. 190, and the method of determining the centre by geometrical construction has been explained in Art. 189.

The centre of a system of parallel forces may also be found by an algebraical calculation. The algebraical method may be considered under three heads:—(1) When the points of application of the forces are in the same line; (2) when they are in the same plane; (3) when they are in different planes.

218. (1) *The magnitudes of a number of parallel forces whose points of application are in the same line, and the distances of these points from a point in the line being given, to find the position of the centre of the forces.*

Prop. 1. *The moment of the resultant round the point is equal to the algebraic sum of the moments of the forces round the same point* (Art. 204).

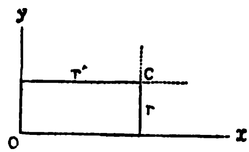


Let P, Q, S, be given parallel forces acting at the points A, B, C, respectively. Let O be any point in the line and let $OA = p$, $OB = q$, $OC = s$. Let the resultant which is $= P + Q + S$ act at some point K; it is required to find OK. Let $OK = r$. From O draw OC′ at right angles to the directions of the forces; then OA, OB, OC, are proportional to OA′, OB′, OC′. But, by Art. 200, $R \cdot OK' = P \cdot OA' + Q \cdot OB' + S \cdot OC'$.

Therefore, $Rr = Pp + Qq + Ss$. And since $R = P + Q + S$, r is determined.

If any of the forces such as Q acts in an opposite direction, then, $Rr = Pp - Qq + Ss$.

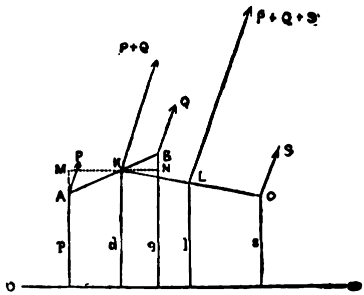
219. (2) *Forces in one plane.*—The algebraical method of finding the centre of a system of parallel forces in one plane consists in determining the distance of the centre from each of two fixed straight lines in the plane; and then the position of the centre is the intersection of two lines drawn parallel to the fixed lines, at the respective distances which have been calculated. It is convenient to take the fixed lines at right angles to each other. Let, for instance, Ox and Oy be two straight lines at right angles to each other in the plane containing the system of parallel forces, and let the distance r of the centre of the system from Ox be found, and the distance r' of the centre Oy be found; then drawing lines parallel to Ox and Oy at distances r and r' respectively, the intersection of these lines C is the centre. The perpendiculars r and r' are called co-ordinates of the point C, and the lines Ox and Oy are the axes of the co-ordinates.



In order to calculate r and r', the distances of the points of application of each of the forces of the system from the lines Ox and Oy must be known. The distances r and r' may then be determined by the following proposition.

220. ***The magnitudes of a number of parallel forces in one plane being given, and the distances of their points of application respectively from a straight line in the plane, to find the distance of the centre of the system from the same straight line.***

Prop. 2. The moment of the resultant of any number of parallel forces in one plane with respect to any line in the plane is equal to the algebraic sum of the moments of the forces with respect to the same line.



Let P and Q be two of the forces of the system, and A and B their points of application. Divide A B in K so that P : Q : : BK : AK, then, (Art. 185) K is the point of application of the resultant of P and Q. Let p be the distance of P from Ox, and q the distance of Q; it is required to determine the distance d of the resultant from the same line. Through K draw MKN parallel to Ox and produce DA to M. Then the triangles MKA, BKN are similar, and therefore—

$\frac{\text{BN}}{\text{AM}}=\frac{\text{BK}}{\text{AK}}$. But by construction $\frac{\text{P}}{\text{Q}}=\frac{\text{BK}}{\text{AK}} \therefore \frac{\text{P}}{\text{Q}}=\frac{\text{BN}}{\text{AM}}=\frac{q-d}{d-p} \therefore$

$$\text{P}\ (d-p)=\text{Q}\ (q-d) \therefore \text{P}d-\text{P}p=\text{Q}q-\text{Q}d \therefore d=\frac{\text{P}p+\text{Q}q}{\text{P}+\text{Q}}.$$

In the foregoing the forces are supposed to be *like*. If unlike, we arrive by a similar demonstration at the result,

$$d=\frac{\text{P}p-\text{Q}q}{\text{P}+\text{Q}}.$$

This result can also be obtained by changing the sign of Q in the value of d for *like* forces.

Let C be the point of application of a third force S, and s its distance from Ox. Join KC and divide it at L so that P + Q : S : : CL : KL. Then L is the centre of P + Q acting at K and of S at C. Let l denote the distance of L from Ox. Then by the previous demonstration

$$l=\frac{(\text{P}+\text{Q})d+\text{S}s}{(\text{P}+\text{Q})+\text{S}};$$ and, substituting the value of d,

$$l=\frac{\text{P}p+\text{Q}q+\text{S}s}{\text{P}+\text{Q}+\text{S}},$$ and $(\text{P}+\text{Q}+\text{S})\,l=\text{P}p+\text{Q}q+\text{S}s$.

Where l is the distance of the centre of the forces P, Q, and S from Ox.

In a similar way we may find the distance from Ox of the centre of any number of parallel forces. If the forces be P, Q, S, T, &c., their resultant $R = P + Q + S + T + \&c.$ If r be the distance of the point of application of the resultant R from Ox, then as before,

$$Rr = Pp + Qq + Ss + Tt + \&c.$$

$$\text{and } r = \frac{Pp + Qq + Ss + Tt + \&c.}{P + Q + S + T + \&c.}$$

If any of the forces such as S acts in an opposite direction to the others, the value for r is obtained from above by making Ss negative.

221. Hence, when the forces and their moments with respect to any line such as Ox are given, we can calculate the distance r of the centre of the forces from the same line. In a similar way, the distance r' of the centre from another line Oy can be found. Then the position of the centre is determined by Art. 219.

222. (3) *Forces not in the same plane.*—When parallel forces are not in the same plane, the position of their centre may be determined by finding its distance from each of three planes at right angles to each other. The planes parallel to these and at the distances respectively that have been calculated, will intersect in a point which is the centre of the system of parallel forces. Knowing the magnitudes of the forces, and their distances respectively from any plane, the distance of the centre from the plane is determined by the following proposition.

Prop. 3. *The algebraic sum of the moments of any number of parallel forces with respect to any plane, is equal to the moment of their resultant with respect to the same plane.* This may be established in the same way as the proposition of Art. 220.

Hence, if there be any number of parallel forces P, Q, S, T, &c., and if p, p', p'' be the distances respectively of the point of application of P from each of three planes at right angles to one another, q, q', q'' the distances of Q from the same planes, &c. Let r, r', r'' be the distances of the centre

of the system from the planes respectively, then the following equations are the algebraical expression of the above proposition :—

$$Rr = Pp + Qq + Ss + Tt + \&c.$$
$$Rr' = Pp' + Qq' + Ss' + Tt' + \&c.$$
$$Rr'' = Pp'' + Qq'' + Ss'' + Tt'' + \&c.$$

Hence, there are only three unknown quantities r, r', r'' and there are three independent equations to determine them. The solution is thus always possible. Again, as these are simple equations, the quantities r, r', r'' have each only one value. There is only *one* centre of the forces, and it is the point of intersection of three planes drawn parallel to the planes from which moments are taken, and at distances r, r', r'' respectively from these planes. We have thus another proof of the statement made in Art. 190, that a system of parallel forces has always a centre, and has only one centre.

Centre of Gravity.

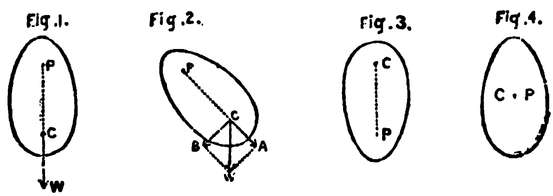
223. *Centre of Mass or Centre of Inertia.*—If the particles of a body be acted on by parallel forces, each of which is proportional to the mass of the particle on which it acts, then the centre of this system of parallel forces may be called the Centre of Mass or the Centre of Inertia of the body.

224. The particles of material bodies are acted on by forces which very nearly fulfil the above conditions. The earth exerts an attractive force upon every particle of a body, and this force is proportional to the mass of the particle. The direction of the force is to a point very nearly coincident with the earth's centre. If the earth were a homogeneous sphere, and at rest, the directions of all the forces of the attraction of gravitation on the particles of a body would intersect in the centre of the earth. This point is at such a distance from a body on the earth's surface that these directions may be regarded as parallel. Two plumb lines suspended near each other, though really in directions that intersect near the earth's centre, are sensibly parallel. Hence the

particles of a body may be considered as acted upon by a system of *parallel forces* due to gravity. The centre of this system of forces is the centre of gravity. And since each particle is attracted with a force proportional to its mass, and since the forces may be regarded as parallel, the *centre of gravity*, according to this assumption, will coincide with the *centre of inertia* of the body. In what follows we shall regard these two centres as coinciding, and we shall make use of the propositions which have been demonstrated respecting the centre of parallel forces for the determination of the centre of gravity of a body or system of bodies.

225. *Centre of Gravity.*—The centre of gravity of a body or system of bodies is then the point through which the resultant of all the forces due to the earth's attraction of the body or system passes, no matter what may be the position of the body or system (Art. 190). The sum of all these forces is the *weight* of the body or system of bodies. The weight always acts through this centre; hence the centre of gravity may be also defined as follows:—The C. G. of a body or system is that point which being supported, the body or system will remain at rest in every position, if all its parts be rigidly connected with that point and with one another.

226. Since the weight of a body is a force always acting vertically through its C. G., it follows that if a body be suspended on an axis round which it can freely turn, it will remain at rest only when the C. G. and point of suspension are in the same vertical line.



If in Fig. 1, the centre of gravity C lie vertically under the point of suspension P, the weight W of the body is counteracted by the reaction of the fixed point, and the body

remains at rest. If the body be moved to the position of Fig. 2, it will not remain at rest. For the force W may be resolved into the two forces CA and CB, CA is met by the reaction of P; CB is not counteracted, and therefore the body will move in the direction of this force, and after some oscillations it will come to rest as in Fig. 1. If the C. G. be vertically over the point of suspension, as in Fig. 3, the weight is met by the reaction at P, and the body will be at rest. If, however, in this case it suffer the slightest displacement, the force of gravity will cause the body to swing round the axis of suspension until at length the body comes to rest as in Fig. 1. If, finally, the point of suspension coincide with the C. G., then the body will rest in any position, since its weight is in every position counterbalanced by the reaction of the fixed point.

There are, therefore, three states of equilibrium:—Stable, Unstable, and Neutral. These are represented respectively in Figs. 1, 3, and 4. If a body be suspended and in *stable* equilibrium, the C. G. is below the point of suspension, and if the body suffer a slight displacement the force of gravity tends to bring it back to its former position. In *unstable* equilibrium the C. G. is above the point of suspension, and if the body suffer a slight displacement, the force of gravity tends to bring the body away still further from its first position. In *neutral* equilibrium the C. G. coincides with the point of suspension, and if the body suffer a displacement the force of gravity has no tendency to change its new position.

227. It will be observed in the above cases that when the body is in stable equilibrium, a slight displacement *raises* the C. G.; when in unstable, *lowers* it; and when in neutral equilibrium neither raises nor lowers it. The alteration in the height of the C. G. may then be adopted as the test of the kind of equilibrium for any body, whether suspended or not. For example, a cone resting on a horizontal plane on its base, is in stable equilibrium; resting on its apex is in unstable equilibrium; and resting on its side is in neutral equilibrium. In the first position a slight displacement

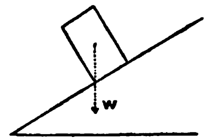
raises the C. G., in the second lowers it, and in the third leaves it at the same height above the plane.

228. A body placed on a plane will stand or fall according as its C. G. falls within or without the outline of the base on which it rests. When a body, such as a book, rests upon a table, the sum of the weights of the particles is equivalent to the weight of the whole book acting vertically downwards through its C. G.; and the reactions of the plane at the indefinite number of points where the book touches the plane, are equivalent to an upward force equal to the weight of the book, and acting vertically upwards through its C. G.

The body may be regarded, therefore, as acted upon by two forces, equal and opposite, and it is consequently in equilibrium. If the body touch the plane at some portions only of its base, as in the case of a table resting on the floor, then, as before, the reactions upwards through the legs of the table are equivalent to one single upward force, which acts through the C. G. of the table, and which is equal and opposite to its weight. It is evident that the direction of the resultant of all the reactions must fall within the outline of the base on which the body rests, and, consequently, if equilibrium be maintained, the direction of the weight of the body acting vertically downwards through the C. G., must also fall within this outline. If, for instance, a uniform circular table rests by means of four legs upon a horizontal plane, the weight acts downwards through the centre of the table, and the resultant of the upward reactions passes through the same point, and the direction of both falls within the area of the base, which may be determined by drawing a string round the feet of the table. If, however, a heavy weight were placed near the edge of the table, the C. G. may be so altered that the vertical through it will fall without the base. The resultant of the reactions must act from some point within the base, and consequently cannot equilibrate the weight. Motion must therefore take place, and the table will turn over.

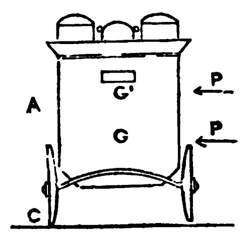
If a body rest upon a plane, and the latter be tilted up,

the direction of the vertical through the C. G. approaches more and more nearly the boundary of the base as the inclination is made greater. In the figure, the vertical passes through the boundary of the base, and the body therefore still remains at rest. Any further inclination will cause it to topple over.



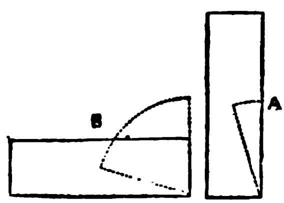
229. Stability of Bodies.—*The lower the C. G. of a body the greater its stability, the base and the weight remaining the same.*

Fig. 7.



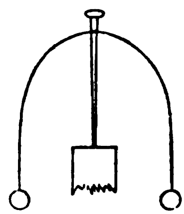
Let A be a coach whose C. G. is G., and let a force P act horizontally through G. This force tends to produce rotation round C. Let now some luggage in the body of the coach be piled on the top, and let the new C. G. be G′. Then the same force P will, for two reasons, have a greater effect in overturning the body. First, the moment of P round C is greater than before. Secondly, G′ will have to be raised to a less height and through a smaller angle than G in order to overturn the body.

The wider the base the greater the stability. A body in the position A is more easily overturned than in the position B, for its C. G. in the first position has to be raised only through a comparatively small distance till it comes into the vertical line through the edge of the base.



230. Hence, it will be seen why a cart with a high load of hay is more likely to be overturned on a rough road than

one with an equal weight of some heavy material. A small boat also is in danger of being upset when the passengers suddenly stand up. A pointed body, such as a pin, cannot remain standing by itself upon a smooth hard surface. The base is so small that the slightest movement of the air causes the C. G. to be unsupported. If, however, two wires be attached to the pin, one on each side, with heavy balls at the ends which are lower than the point of support (as in the figure), the C. G. of the whole will be below the point of suspension, the system will be in stable equilibrium, and the pin can oscillate on its point without falling.



If a man carries a load on his back he leans forward, if he supports a weight in his arms he leans backwards. A man leans forwards when ascending a hill, and backwards when descending. In all such cases the man takes such a position that the vertical through the C. G. of his body falls within the base of support.

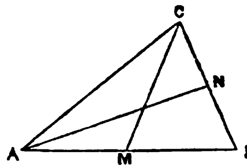
231. *Experimental determination of the C. G. of a body.*— The principle explained in Art. 226, will enable us in some cases to determine by experiment the C. G. of a body. If the body be suspended on an axis round which it can freely turn, or if it hang by a flexible string, the C. G. lies in the vertical line drawn through the point of suspension. If this line be marked, and the body be suspended from another point, a new line containing the C. G. may similarly be found. And since the C. G. lies in each line, it must be at their point of intersection. In this way the C. G. of a material surface or of an open framework may be approximately determined.

When we speak of finding the C. G. of a line or of a surface, it must be understood that we are speaking not of geometrical but of material lines and surfaces. A material line is an indefinitely thin rod, and a material surface is an indefinitely thin slice or lamina.

232. *Determination of the C. G. of a body or a system of*

238. *To find the C. G. of a triangle.*

We may suppose the triangular lamina made up of an indefinite number of lines parallel to one side AB. Bisect AB at M and join CM. It can be easily shown by geometry that CM bisects all the lines parallel to AB; therefore the C. G. of the triangle lies in CM. Similarly the C. G. lies in the line AN bisecting another side BC. Therefore the C. G. of the triangle is at the point of intersection O of the lines CM and AN.



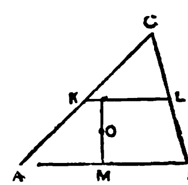
By a well-known geometrical theorem each of the lines CM and AN is divided at O, so that one part is double the other. OM is consequently one-third of CM. Hence the C. G. of a triangle lies on the line joining the middle point of any side with the opposite angle, and at one-third of the length of this line from the side.

239. *The C. G. of a triangle coincides with the C. G. of three equal heavy particles placed at the angular points of the triangle.*

In the triangle ABC (Fig. of last Art.) let three equal particles be placed at the points ABC. Then the C. G. of the particles at A and B is at M. Join M with C and divide MC into two parts at the point G so that MG is half of GC. Then G is the C. G. of the three particles, and MG is one-third of MC. Therefore the point G coincides with the point O.

240. *To find the C. G. of the perimeter of a triangle.*

Let AB, BC, CA be uniform thin rods enclosing a triangular space. Bisect these lines in the points M, L, K. The weights of the lines may be regarded as acting at these points respectively. Join KL and divide it N inversely as the weights of AC and CB. Join NM and divide it at O, inversely as the weights of AC + CB and AB. Then O is the C. G. of the perimeter.

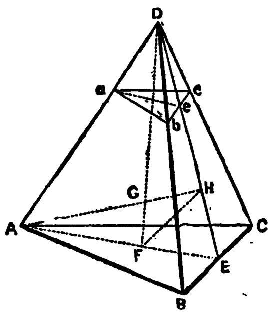


It may be shown by geometry that O is the centre of the circle inscribed in the triangle KML.

241. *To find the C. G. of any plane rectilineal figure.*

Divide the figure into triangles, and find (Art. 238) the C. G. of each triangle. The weights of the triangles respectively may be regarded as acting at these points, and by applying Art. 233 we can find the C. G. of the whole.

[*] 242. *To find the C. G. of a triangular pyramid.*



Let ABCD be the pyramid. Bisect BC in E; join AE; take $EF = \frac{1}{3}AE$; then F is the C. G. of the triangle ABC (Art. 238).

Join FD; then the C. G. of the pyramid lies in the line FD. For the pyramid may be supposed to be made up of an indefinite number of triangles parallel to ABC, and FD passes through the C. G. of each of these triangles. For let $a\,b\,c$ be one of these triangles. Join ED. ED bisects bc (see Art. 238). Join $e\,a$; then the C. G. of $a\,b\,c$ lies in $a\,e$. But $a\,e$ is a line in the plane DAE, and DF is another line in this plane, therefore DF intersects $a\,e$ in some point f. From similar triangles (Euc. VI. 4) DF : Df : : AF : af and DF : Df : : FE : fe ∴ (Euc. V. 11, and alternately) AF : FE : : af : fe ∴ $af = 2fe$ ∴ f is the C. G. of $a\,b\,c$. Hence the C. G. of the pyramid lies in DF. Similarly by taking $EH = \frac{1}{3}ED$ and joining HA we may show that the C. G. of the pyramid lies in HA. Therefore the point of intersection G of HA and DF in the plane DEA is the C. G. of the pyramid.

By similar triangles in the plane DEA,

$$AD : FH :: AE : FE :: 3 : 1 \therefore AD = 3FH.$$

Again in the triangles—

$$\text{ADG, FGH (Euc. VI. 4)}\ \frac{FG}{GD} = \frac{FH}{AD} = \frac{1}{3}.$$

$$\therefore GD = 3FG \text{ and } DF = 4FG.$$

Hence the C. G. lies in the line FD at $\frac{1}{4}$th of its length from the base.

243. *The C. G. of a triangular pyramid coincides with the C. G. of four equal heavy particles placed at the angular points of the pyramid.*

Let four equal heavy particles be placed at the points A, B, C, D (see Fig. of last Art.) Then (Art. 239), the C. G. of the particles at A, B and C is F. Join FD and divide it inversely as the sum of the weights of the particles at A, B, C to the weight of the particle at D. Hence FD will be divided in the ratio 1 : 3, and therefore the point of division which is the C. G. of the four particles will coincide with G.

244. *To find the C. G. of any Pyramid having a plane rectilineal polygon for base.*

The base may be divided into triangles, and the given pyramid will be equivalent to as many triangular pyramids as there are triangles in the base. The C. G. of each of these triangular pyramids will be on the line joining the C. G of its base with the vertex at one-fourth its length from the base. Therefore the C. G. of the whole pyramid will be in a plane parallel to the base and at one-fourth the perpendicular distance of the vertex from the base, since this plane will pass through the C. G. of each of the pyramids. But we may suppose the pyramid to be made up of an indefinite number of laminæ parallel to the base. The C. G. of each of these will lie on the line joining the C. G. of the base with the vertex; therefore the C. G. of the pyramid lies on this line. Hence the C. G. of the pyramid is the point where this line intersects the plane parallel to the base, and is therefore at one-fourth of the length of this line from the base.

245. *To find the C. G. of a Cone.*

Since a circle may be regarded as a polygon having an infinite number of sides, a cone may be regarded as a pyramid having such a polygon for its base. The C. G. of a

cone lies therefore on the line joining the centre of the base with the vertex, at one-fourth its length from the base.

246. *Algebraical method of determining the Centre of Gravity.*

The C. G. of a system of particles may also be determined by the principles explained in Arts. 218–222.

If the particles lie in the same straight line, and if their weights and their distances respectively from a point in the line be known, then the sum of the moments of the particles round this point will be equal to the moment of their resultant round the same point. Hence the distance of the C. G. from this point may be found.

247. If the particles lie in the same plane, and their weights and their distances respectively from each of two lines in the plane be known, the distance of their C. G. from each line can be calculated by the following proposition (see Art. 220).

The sum of the moments of the weights with respect to any line is equal to the moment of their resultant with respect to the same line.

Hence the position of the C. G. is found as in Art. 221.

248. If the particles are not in the same plane, their weights and their distances respectively from each of three planes being given, then the position of the C. G. is determined as in Art. 222.

It is convenient in the above cases to have the lines and the planes, with respect to which the moments are taken, at right angles to each other.

249. *Principle of Symmetry.*—We can frequently determine at once the C. G. of certain bodies by the principle of symmetry. A body is said to be symmetrical with respect to a plane when it may be regarded as f rmed of equal particles equidistant from the plane. The C. G. of the body must evidently lie in this plane. If the body be symmetrical with regard to two or more planes, the C. G. must lie in the line or point of intersection of the planes.

Applying this principle to the following bodies, we see at once that the C. G. of a—

Line . . .	Is its	middle point.
Parallelogram .	,,	point of intersection of diagonals.
Perimeter of do. .	,,	do. do.
Circle . . .	,,	centre.
Circumference of do.	,,	do.
Parallelopiped .	,,	intersection of diagonals.
Cylinder . .	,,	middle point of axis.
Sphere . .	,,	centre.

Examples.

1. A uniform pencil rests on a table with $\frac{5}{12}$ths of its length projecting beyond the edge. A beetle whose weight is $\frac{1}{4}$th that of the pencil crawls along it: how far may it crawl without upsetting the pencil?

Let w = weight of pencil, and x = the distance of the beetle beyond the edge of the table when the pencil is on the point of being upset. The resultant then passes through the edge of the table. The weight of the pencil acts at its middle point, which is $\frac{1}{12}$th of its length from the edge. Taking moments round the edge $\therefore$ (Arts. 202 and 246),

$$\tfrac{1}{4}Wx = W\tfrac{1}{12} \therefore x = \tfrac{1}{3}.$$

Therefore the distance is $\frac{1}{3}$rd of the length of pencil from the edge, or $\frac{1}{12}$th from the end.

2. A bar supposed to be without weight is 5 feet long, and has weights of 1, 2, 3, and 4 lbs. suspended at the distances respectively of 1, 2, 3, and 4 feet from one extremity: where must a fulcrum be placed so that the bar may balance upon it?

Let x = the distance of the fulcrum from the extremity. Taking moments round that extremity $\therefore$ Art. 246,

$$10x = 1\times1+2\times2+3\times3+4\times4 \therefore x = 3 \text{ ft. from end.}$$

3. A circle is described upon one of the radii of another circle, and the smaller circle cut out. Find the C. G. of the remainder, the diameter of the larger circle being $2a$.

Circles are proportional to the squares of their diameters; therefore the circles are as $4a^2 : a^2 \therefore 3a^2$ is proportional to the area of the remainder. Let x = the distance of its C. G. from the centre of the larger circle. Taking moments round the centre $\therefore$ (Art. 202),

$$3a^2 \times x = a^2 \times \frac{a}{2} \therefore x = \frac{a}{6}.$$

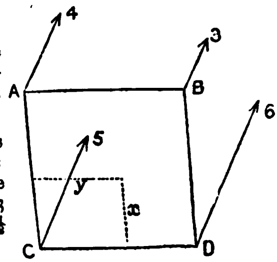
4. Five equal bodies are placed at five of the angles of a regular hexagon whose side is a: find the distance of their C. G. from the unoccupied corner.

The C. G. will evidently be upon the diagonal passing through the unoccupied angle.

Let x = the distance required, and let w = weight of each body. Taking moments round the unoccupied angle (Art. 246) thus,

$$5w \times x = 2w \times \tfrac{1}{2}a + 2w \times 1\tfrac{1}{2}a + w \times 2a \therefore x = 1\tfrac{1}{5}a.$$

5. Parallel forces of 3, 4, 5, and 6 lbs. act respectively at the corners of a square each side of which is 3 feet: find the distance of the centre of these forces from two adjacent sides of the square.



We employ Art. 247. Take moments with respect to CD, $\therefore 3 \times 3 + 4 \times 3 = (3 + 4 + 5 + 6)x \therefore x = 1\tfrac{1}{6}$ feet. Take moments with respect to AC, $\therefore 3 \times 3 + 6 \times 3 = (3 + 4 + 5 + 6)y \therefore y = 1\tfrac{1}{2}$ feet.

Therefore the centre of the forces is $1\tfrac{1}{6}$ ft. from CD, and $1\tfrac{1}{2}$ ft. from AC.

Exercises.

1. A weight is suspended by a string from a hook: show that it will remain at rest only when its centre of gravity is vertically below the point of suspension.

2. A man when going up a hill leans forwards, but when coming down leans backwards: why? A horse when drawing a heavy load up a hill takes a zigzag course: why?

3. Explain what is meant by "stable," "unstable," and "neutral" equilibrium. Give an example of a body in each of these conditions.

4. A uniform wire is bent so as to form the perimeter of a triangle: show how the C. G. of this perimeter may be found.

5. Show how to determine the C. G. of an indefinitely thin triangular plate:—

(*a*.) By a geometrical construction.

(*b*.) By experiment.

6. Particles whose weights are 2, 2, and 1, are placed at the corners of an equilateral triangle whose side is 10 feet: how far is the C. G. from the smaller particle?

7. A tower is built in the form of an oblique cylinder upon a base whose diameter is 60 feet. The inclination is such that for a slant height of 5 feet there is a vertical height of 4 feet. What is the greatest vertical height such a tower can have?

8. A circular table rests on three legs placed at equal distances on the circumference. If the weight of the table be 20 lbs., what is the greatest weight which may be placed on any point of the table without upsetting it?

9. A triangular slab of uniform thickness weighing 90 lbs. is supported in a horizontal position by three props at the corners: what is the pressure on each prop?

10. Four particles whose weights are 3, 5, 7, and 9 are placed at the corners A, B, C, D of a square the side of which is one foot: find the C. G. of the particles.

11. A beam 16 feet long and weighing 50 lbs. is thicker at one end than at the other, its C. G. lying at the distance of 5 feet from the thicker end. This end rests on a prop: where must another prop be placed so that the pressures on the props may be equal, the beam being horizontal?

12. Four bodies weighing 1, 2, 4, and 7 lbs. respectively are placed with their centres of gravity in a straight line at the respective distances of 3, 5, 7, and 9 feet from one extremity of the line: find the C. G. of the weights.

13. A bar each foot of which weighs 7 lbs. rests upon a fulcrum distant 3 feet from one extremity: what must be its length that a weight of $71\frac{1}{2}$ lbs. suspended from that extremity may just be balanced by 20 lbs. suspended from the other extremity?

14. The top of a triangle is cut off by a straight line parallel to its base, and at a distance from it of two-thirds of the height of the triangle: find the C. G. of the remaining part.

15. Two books similar in every respect each 10 inches long, lie one exactly over the other on a table, over the edge of which they project 3 inches. How much further may the upper book be pushed out before they fall over?

16. Seven bodies of equal weight are placed so that their centres of gravity coincide with as many angles of a cube, the diagonal of which is a: find the distance of their C. G. from the unoccupied angle of the cube.

17. A straight line parallel to the base of a triangle, cuts off a triangle which is one-fourth of the whole: find the C. G. of remainder.

18. From a material circle another circle described on a radius is cut out: find C. G. of remainder.

19. Three men at the corners of a heavy slab support it: show that each bears an equal pressure.

20. A heavy uniform rod 40 inches long is bent at its middle point so that the arms are at right angles. Find the distance of the new position of the C. G. of the rod from its position before being bent.

21. A uniform wire is bent so as to form the perimeter of a right angled triangle, the lengths of the sides being 10, 8, and 6 feet. Find the distance of the C. G. of the perimeter from each of the latter sides.

22. A square whose side is 10 has an equilateral triangle described on one side: find the C. G. of the whole figure.

23. Determine by a geometrical construction the C. G. of half of a regular hexagon.

24. Particles whose weights are 1 and 7 are placed at the ends respectively of a rod 2 ft. long, the weight of which is neglected, and particles whose weights are 3 and 5 are placed each at a distance of 8 inches from

the weights 1 and 7 respectively. Find the distance of the C. G. from the middle point of the rod.

25. The side of a square is 10 ft. One of the triangles formed by the diagonals is cut out: find the C. G. of the remainder.

26. The diagonal of a square is 10 feet. Two lines drawn through the middle points of the opposite sides divide the square into four smaller squares. One of these is cut out: find the C. G. of remainder.

27. From a square whose side is 4 inches, a corner square whose side is 1 inch is cut out: find C. G. of remainder.

28. Find the C. G. of the surface of a right cone.

29. Two uniform cylinders of the same material, one of them 8 inches long and 2 inches in diameter, the other 6 inches long and 3 inches in diameter are joined together end to end so that their axes are in the same straight line. Find the C. G. of the combination.

30. Weights of 1, 2, 3, 4, 5, 6, 7, and 8 lbs. are placed in order at the corners of an octagon whose side is a. The centres of gravity of the weights coincide with the angular points respectively. Find the distance of the C. G. of the whole of the weights from the centre of the octagon.

31. Weights of 1, 2, 3, 4, 5, and 6 lbs. are placed in order round the corners of a regular hexagon whose side is a: find the distance of C. G. of weights from the centre of the hexagon.

32. One of the triangles formed by the intersection of the diagonals of a square whose side is a is taken away: find the C. G. of the remainder.

33. From a cone whose height is 4 inches, another cone 2 inches in height is cut off by a plane parallel to the base: find the C. G. of the frustum.

34. A uniform beam 25 feet long and weighing 100 lbs. has its upper end resting against a smooth vertical wall, and its lower on a smooth floor at the distance of 7 feet from the wall, this end being connected with the base of the wall by a string: find the tension in the string.

35. Four spheres whose weights are 1, 3, 5, and 9 lbs. are placed so that their centres coincide with the angular points of a square whose side is 4 feet: find their C. G.

CHAPTER XVI.

MACHINES.

250. When a force does work at one point, an equal amount of work may be obtained at another assigned point. A machine is an instrument by which the energy available at any place can be transferred to another place. The force applied to a machine is usually called the Power, and when this force does work by the expenditure of a certain amount of energy, an equal amount of work is done against resisting forces. These resisting forces may be a weight which is to be lifted, and the friction of the parts of the machine. Neglecting for the present the forces of friction, the work done by the power is equal to the work done against the weight. Calling P the power and p the distance through which it moves its point of application in the direction in which it acts, R the weight and r the distance through which it moves, then, Art. 135,

$$Pp = Rr.$$

251. The construction and action of all machines may be referred to one or more of the Simple Machines or Mechanical Powers, which are six in number. They may be arranged in three classes, as follows :—

I.—(1) The Lever ; (2) The Wheel and Axle ;
II.—(3) The Inclined Plane ; (4) The Wedge ; (5) The Screw ;
III.—(6) The Pulley.

252. We shall investigate the conditions of equilibrium for each of these simple machines. The problem in each case may be considered from two points of view. The machine may be *at rest* under the action of the equilibrating forces of the power and weight, or it may be moving *uniformly* under the same equilibrating forces. Hence the problem

may be examined either statically or kinetically. We shall adopt both methods; and for the present we shall assume that the machine acts without friction.

When there is equilibrium with any machine, the ratio of the Weight to the Power is called the *Mechanical Advantage* of the machine.

THE LEVER.

253. The Lever is a rod straight or bent, turning round a fixed point called the *fulcrum*, and having at least two forces, usually called the *power* and the *weight*, acting at other points of the bar. This rod is supposed to be inflexible.

254. Levers are usually divided into three classes according to the relative positions of the fulcrum, the weight, and the power. When the fulcrum is between the power and the weight, the lever is of the first class; when the weight is between the power and the fulcrum, the lever is of the second class; and when the power is between the weight and fulcrum, it is of the third class.

The following are examples of the first class of levers:—The handle of a pump; a poker resting on the bar of a grate. To the second class belong—The oar of a boat; a chipping knife with a hinge at one end. To the third class—The treadle of a turning lathe. Examples of double levers of each class are—A pair of scissors; a pair of nutcrackers; a pair of tongs.

In the human body there is an example of each class of lever. When a man standing on his feet raises his body so as to rest on his toes, his feet are levers of the first class, the fulcrum being at the ancle, the power applied at the heel by the muscle attached to the leg, and the weight acting vertically at the toes. The jaws form a double lever of the second class. The forearm is a lever of the third class when a weight placed in the hand is raised by the action of a muscle attached to a point near the elbow, the latter being the fulcrum.

This distinction into classes of levers is not of any

theoretical importance. The same rule applies to all kinds of levers.

255. The ratio between the power and the weight when there is equilibrium with a lever may be determined by statical principles as follows :—

When there is equilibrium the resultant of the power and weight passes through the fixed point or fulcrum, the reaction of which is equal to the resultant pressure upon it. Thus the forces acting on the lever equilibrate each other, and it remains at rest.

Fig. 1.

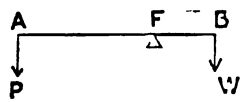


Fig. 2.

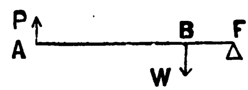


Fig. 3.

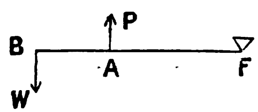
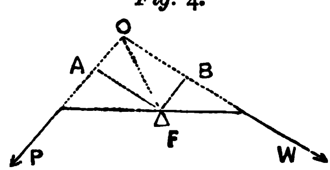


Fig. 4.

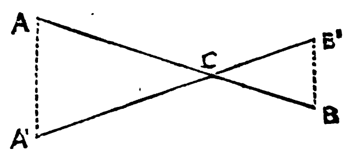


In Figs. 1, 2, 3 the forces are at right angles to the lever; in Fig. 4 they are oblique. In Fig. 1 the resultant is equal to the sum of P and W; in Figs 2 and 3 equal to their difference; in Fig. 4 it is represented by the diagonal OF of the parallelogram AB where OA and OB are proportional to P and W.

In all of these cases, when P and W are in equilibrium their resultant passes through F, and the moments of P and W round F are equal and opposite (Arts. 198, 202). Therefore $P \times FA = W \times FB$. That is P *multiplied by the perpendicular from* F *on the direction of* P = W *multiplied by the perpendicular from* F *on the direction of* W. Calling these perpendiculars the arms of P and W, respectively, the rule is frequently expressed as follows :—

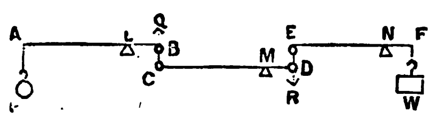
$$P \times \text{arm of } P = W \times \text{arm of } W.$$

256. This rule applies to all classes of levers. The distinction into classes is of use in determining the arms of the power and weight respectively. In the third class of levers the arm of the power is less than the arm of the weight; the power must therefore be greater than the weight, and there is consequently a *mechanical disadvantage*, but a corresponding gain in velocity.



257. The rule for the lever may also be determined by the Principle of Work (Art. 135) as follows:—Let a force P act at the point A of the lever AB, and a resistance W at B. Let AB move into the position A′B′ round the fulcrum C, and for the sake of simplicity let AA′ BB′ be vertical lines; then (Art. 135) $P \times AA' = W \times BB'$. But $AA' : BB' :: AC : CB$ (Euc. VI. 4) $\therefore P \times AC = W \times BC$. And the same result may be obtained similarly for the other classes of levers.

258. *Combination of Levers.*—Levers may be used in combination, and thus in a compact form a great mechanical advantage may be produced.



The levers AB, CD, EF, having their fulcrums at L, M, and N, may be connected at the ends B, C and E, D, so as to form one system. P acting downwards at A produces an upward force at B, and this causes a downward force at D. Let Q be the force at B, and R that at D. Then if the levers are in equilibrium (Art. 255) $P \cdot AL = Q \cdot BL$; $Q \cdot CM$

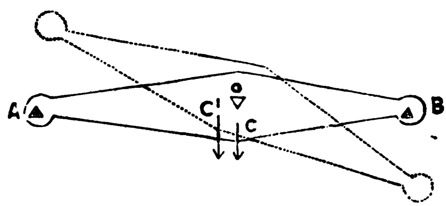
$= R \cdot DM$; $R \cdot EN = W \cdot FN$. Multiplying the corresponding terms of these equations, we obtain

$$P \cdot AL \cdot CM \cdot EN = W \cdot BL \cdot DM \cdot FN.$$

That is the power × product of arms of power = weight × product of arms of weight.

259. As examples of the lever of great practical importance, we shall here consider at more length than we can devote to other instruments some forms of the Balance.

The Common Balance.—The common balance is a lever of the first kind. As it is used to determine the equality of the two weights placed in the scales, the arms are made equal. It consists of a beam turning round a fulcrum which is equidistant from the two ends, to which scale pans are attached.



The requisites of a good balance are:—(1) Accuracy, (2) Stability, (3) Sensibility.

A *true* or *accurate* balance rests with its beam horizontal when equal weights are placed in the scale pans. A *stable* balance returns to the horizontal position if disturbed. A *sensible* balance turns when a very slight additional weight is added to one scale.

260. The *accuracy* of a balance depends on the following condition:—

The arms should be of equal length.

The balance is said to be *true* if the arms be equal, and *false* if unequal. The arms must remain unaltered in length during the oscillations of the beam. This is effected by attaching to the beam a triangular prism with its edge downwards—called a *knife edge*—and the beam rests by this

edge on its supports. The axis of suspension is thus a mere line, and during the oscillations of the beam the lengths of its arms are unaltered. In the finest balances constructed for physical and chemical researches the beam rests by a knife edge of tempered steel upon an agate plane, and the scales also rest by means of agate planes on knife edges of steel turned upwards at the ends of the beam.

If O be the fulcrum and C the C. G. of the beam and scales, then when equal weights are hanging from A and B, the beam will remain horizontal with C directly under O, since the arms being equal the moments round O are equal and opposite.

If when equal weights are in the scales, one of the scale pans is pushed inwards or outwards, the moments round the fulcrum are no longer equal, and the beam will not remain horizontal.

261. The *stability* of the balance depends on the following conditions:—

(*a.*) *The C. G. of the beam and scales should be below the axis of suspension.*

If the C. G. of the balance be in the axis of suspension, the beam will remain at rest in any position, and thus the horizontality of the beam cannot be used to determine the equality of the weights in the scales. If the C. G. be above the axis of suspension, the equilibrium will be unstable (Art. 226), and the slightest movement will cause the beam to overturn. The C. G. should therefore be below the axis. If, then, the beam be turned into any position such as that indicated by the dotted lines, the moments of the weights in the scales are still equal, but the C. G. is now at C′, and the weight of the balance tends to turn the beam back to its original position with a moment equal to the product of the weight of the balance into the perpendicular from O on the vertical through C′, and after some oscillations the beam will come to rest with its C. G. vertically under O.

(*b.*) *The axes of suspension of the scales should be in the same plane with the axis of suspension of the beam.*

So far as regards the equilibrium of the beam the C. G. of the weights in each scale may be regarded as coinciding with the point of suspension of the scale pan. Hence if the axes of suspension of the scales be above the axis of the beam, the C. G. of the whole balance becomes raised when weights are placed in the scale pans, and it may by additional loading be made to coincide with the plane of the axis of suspension of the beam, and the balance will then cease to act. If on the other hand the axes of suspension of the scales be under that of the beam, increased loading will lower still further the C. G. of the balance, and will increase its stability, but diminish its sensibility. But if all the axes are in the same plane, increased loading will raise the C. G., which, however, can never rise so high as the axis of suspension. The sensibility will therefore be increased, while the balance will still remain stable.

(c.) *The beam should be rigid.*

Any bending would evidently affect either the stability or the sensibility. Hence the beam is usually made in the form of a very elongated rhombus.

262. The *sensibility* of a balance depends on the following conditions being satisfied in its construction.

(a.) *The friction should be as small as possible.*

The more readily the balance turns, the more clearly it indicates any inequality in the weights. The knife edges diminish friction during the play of the balance, and consequently increase the sensibility.

(b.) *The arms of the beam should be as long as possible.*

The greater is then the moment of a small additional weight in turning the beam.

(c.) *The beam should be as light as possible.*

If the beam be turned by an additional weight in one scale into the position indicated by the dotted lines, the moment tending to turn it back to its original position is the weight of the balance into the perpendicular from O on the vertical through C′. The heavier the beam the greater

this moment, and the greater, therefore, the additional weight required to produce a given deflection.

(*d.*) *The C. G. of the balance should be near the axis of suspension.*

The nearer it is the less the moment which, when the beam is deflected, tends to turn it back again.

Sensibility and stability are therefore qualities of a balance which to some extent are incompatible. Hence the character of a balance must be determined by the use to which it is to be put. For refined physical weighings, *sensibility* is the more important requisite; for ordinary purposes, a more *stable* balance is better adapted.

263. The equality of the arms of a balance may be tested as follows:—A body placed in one scale is equilibrated by weights in the other, and the weights and the body are then transposed. If equilibrium still exists the arms are equal, if not they are unequal. A very slight inequality in the arms is at once detected in this way; for if equilibrium is produced with a false balance, and the bodies transposed, the heavier is now attached to the longer arm and the lighter to the shorter, and the moments must be unequal.

264. With a false balance the true weight of a body may be easily obtained by either of the following methods of double weighing. These methods are frequently employed even with the best chemical balances, since even with the utmost skill and care it is impossible to make the arms exactly equal.

(1) Place the body in one scale of the balance, and equilibrate with some substance such as sand in the other scale. Then take out the body and substitute known weights in its place until once more there is equilibrium. The weights are then the true weight of the body, since acting at the same arm, the weights and the body have each equilibrated the sand.

(2.) Weigh the body first in one scale and then in the other; multiply the apparent weights together, and extract the square root of the product. This is the true weight of the body.

For let a and b be the arms of the false balance, let w be the true weight of the body, let x be the apparent weight when attached to b, and y when attached to a: then (Art. 255),

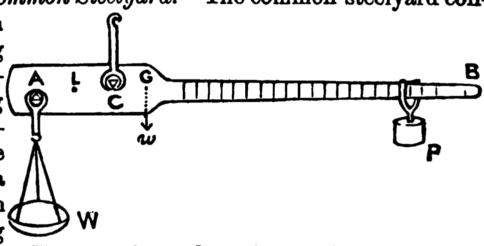
$$wa = bx$$

and $wb = ay$; multiplying these equations,

$$\therefore w^2ab = abxy \therefore w^2 = xy \text{ and } w = \sqrt{xy}.$$

265. A delicate physical balance requires the utmost care in its construction and use. The beam except when in use is not allowed to rest on the knife edges; stops, which can be raised or lowered, are placed under the scale pans; the weights are moved with a forceps, and are not touched by the hand; the whole is enclosed in a glass case which preserves the balance from dust. A long index oscillates with the beam, so as to show a very slight deflection. A delicate balance will turn with the millionth part of the load added to either scale.

266. *The Common Steelyard.*—The common steelyard consists of a beam A B turning round a fulcrum C, having a scale pan attached to one arm and a weight P which can slide along the other arm.



This arm is graduated, and the weight of a body placed in the scale pan is found by moving P to some point on the arm CB until the beam remains at rest in a horizontal position. The graduation of the beam at this point gives the weight of the body.

267. The beam may be graduated experimentally by placing in the scale pan successively 1, 2, 3, &c., units of weight, and finding by trial the successive positions to which P must be moved in order to keep the beam horizontal. These points are marked 1, 2, 3, &c. In a similar way the beam can be graduated for any other unit

of weight. These divisions of the beam may be subdivided so as to express fractions of the unit weight.

268. The successive positions of P, when 1, 2, 3, &c., units of weight are placed in the scale pan, may also be determined by calculation. Let w be the weight of the beam and scale pan and G the C. G., and let K be the position of P when a weight W is placed in the scale. Taking moments round C, (Art. 204),

$$W \cdot AC = w \cdot GC + P \cdot KC.$$

Take LC : GC :: w : P; then $P \cdot LC = w, GC$;

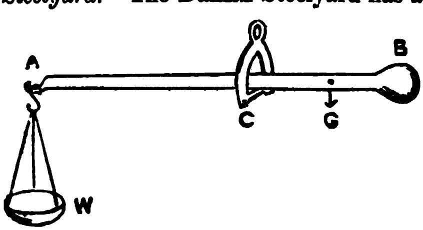
and $W \cdot AC = P \cdot LC + P \cdot KC = P \cdot LK$,

$$\therefore\ LK = \frac{W}{P} \cdot AC.$$

Take W equal successively to P, 2P, 3P, &c., . . . then LK = AC, 2AC, 3AC, &c., ... Therefore measure off from the point L distances equal to AC, 2AC, 3AC, &c., and at these points mark 1, 2, 3, 4, &c.; then when there is equilibrium with P at the points 1, 2, 3, &c., the weights in the scale are respectively P, 2P, 3P, &c. Therefore if P be a known weight, W is also known. If, for instance, P be 1 oz. then the graduation of the beam where P rests when there is equilibrium expresses the number of ozs. in W. The divisions of the beam are all equal, and by subdividing them into equal parts, fractions of the unit weight may be expressed.

Thus with a graduated steelyard the weight of a body may be found without the set of stamped weights required with the ordinary balance.

269. *The Danish Steelyard.*—The Danish Steelyard has a scale pan or hook at one end A, and a heavy knob at the other end, the fulcrum C being movable. When a body W is placed in the scale pan, the beam is pushed through



the ring at C until the weight of the beam and scale pan w acting at the C. G equilibrates W, and then the graduation of the instrument gives the weight of W.

270. The beam may be graduated experimentally in the same way as the common steel yard (Art. 267).

The positions of the fulcrum C for successive units of weight in the scale pan may also be determined as follows:—

Let C be the position of the fulcrum when there is equilibrium with a weight W in the scale pan. Taking moments round C, then

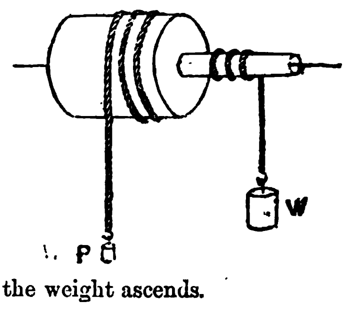
$$W \cdot AC = w \cdot GC = w\,(AG - AC) = w \cdot AG - w\,AC \quad \therefore$$
$$AC = \frac{w \cdot AG}{w + W}.$$

Take $W = w,\ 2w,\ 3w$, &c., . . . then the successive values of AC are $\frac{AG}{2},\ \frac{AG}{3},\ \frac{AG}{4}$, &c., . . . and the position of the fulcrum C for these values of W can be marked on the beam.

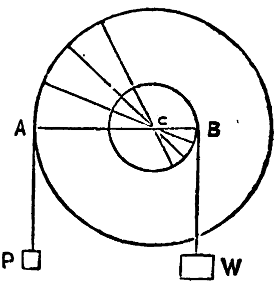
The successive distances of the point C from A as thus determined form an harmonical progression, since the reciprocals of the successive values of AC form an arithmetical progression. These graduations are therefore not equal, and subdivisions of them are made by giving to W intermediate values and calculating the corresponding positions of the fulcrum.

The Wheel and Axle.

271. The wheel and axle in its simplest form consists of two cylinders forming one rigid body, and having a common axis. The larger of these cylinders is called the *Wheel* and the smaller the *Axle*. A cord fastened to the weight is coiled round the axle, and another fastened to the power is coiled in an opposite direction round the wheel, so that as the power descends the weight ascends.



272. The ratio of the power to the weight when there is equilibrium may be determined as follows:—

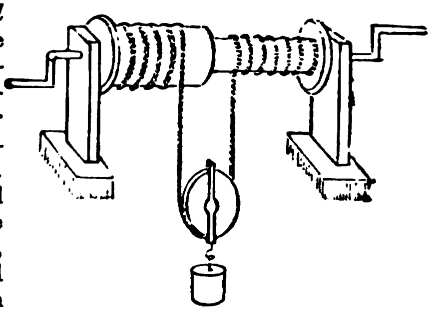


We may suppose P and W to act in the same plane perpendicular to the axis. The figure represents a section of the wheel and axle in this plane. The instrument may be regarded as made up of an indefinite number of levers having a common fulcrum at C, and coming into action successively. Each lever will in turn come into the position ACB where the cords leave the wheel and the axle at right angles to the lever. By this arrangement, therefore, a lever always occupies the position ACB, and by Art. 255, $P \cdot AC = W \cdot BC$. That is, power × radius of wheel = weight × radius of axle. And since the circumferences of circles are proportional to their radii, if C denote the circumference of the wheel, and c that of the axle, $P \cdot C = W \cdot c$.

The same results may be obtained from Art. 135 by supposing the instrument to be in uniform motion. In one turn the power descends through a space C, and the weight ascends through c, therefore by Art. 135 $P \cdot C = W \cdot c$.

273. The wheel and axle assumes many different forms. Instead of a wheel, the power is frequently applied to a handle, the extremity of which describes a circle. The *winch* and the *windlass* are examples. The *capstan* used in ships has its axis vertical, and the power is the sum of the forces with which the men push at the ends of handles inserted into the axle.

274. The mechanical advantage of the wheel and axle $= \frac{W}{P} = \frac{C}{c}$. We can consequently increase the mechanical advantage by making the wheel larger or the axle smaller. This can only be done within certain limits, or the machine will become too unwieldy or too weak. By the following arrangement the mechanical advantage may be increased indefinitely:—



Differential Wheel and Axle.—The axle consists of two cylinders of different diameters. and the power is applied at the extremity of a handle. The rope is coiled round one of the cylinders of the axle, then is passed round a pulley from which the weight hangs, and is then coiled in an opposite direction round the other portion of the axle. Let c, c' be the circumferences of the two parts of the axle, and C the circumference of the circle described by the power. In one turn the work done by the power is $P \cdot C$. The portion of the cord suspended from the axle is shortened in one turn by a length c, and lengthened by a length c', and therefore on the whole it is shortened by a length $c - c'$. The pulley and the weights are consequently raised through $\frac{c-c'}{2}$ (see Art. 299). Therefore (Art. 135) $P \cdot C = W \frac{c-c'}{2}$.

The two circumferences c and c' may be made as nearly equal as we please, and consequently the mechanical advantage may be increased without limit.

275. *Toothed Wheels.*—Toothed wheels are very largely employed in machine work, but the consideration of their construction, the form of their teeth, and their combination, belongs to practical mechanics. When the teeth on the circumference of one wheel work into those of another, the wheels having equal axles, and the number of teeth in each being proportional to the circumference of the wheel, we can apply the rule for the wheel and axle. If there be equilibrium in this case, and if the wheel to which P is attached be called the power-wheel, and that to which W is attached the weight-wheel, then $P \times N^{\circ}$ of teeth in circumference of Power-wheel $= W \times N^{\circ}$ in Weight-wheel.

276. *Combination of Wheels and Axles.*—A number of

toothed-wheels or wheels and axles connected by bands may be combined so as to form one system. The ratio between the Power and Weight when there is equilibrium is expressed by an equation similar to that of Art. 258.

The Inclined Plane.

277. The Inclined Plane is a rigid plane inclined to the horizon. The Power and the Weight are supposed to act in the same vertical plane passing through the inclined plane. The section made by the vertical plane is a right angled triangle, whose hypotenuse is called the *length* of the inclined plane, the perpendicular the *height*, and the horizontal side the *base*. The inclination of the inclined plane is the angle which it makes with the horizon. This angle may be expressed in degrees, or in terms of the ratio of the height to the length. Thus, a plane rising 1 in 2 is one whose height : length : : 1 : 2 ; and therefore its inclination is 30°.

278. When a body is supported on a smooth inclined plane by a power, the three forces acting on the C. G. of the body are in equilibrium. These three forces are the *power*, the *weight* of the body, and the *reaction* of the plane. The relations between these forces may be determined by Art. 174.

There are three cases usually considered :—(1) when the Power acts parallel to the length (Fig. 1) ; (2) when it acts parallel to the base (Fig. 2) ; (3) when it makes a given angle with the perpendicular to the plane (Fig. 3). Let in each case P denote the power, W the weight, and R the reaction. In Fig. 3, let i be the inclination, and θ the angle which the direction of P makes with that of R.

Fig. 1.

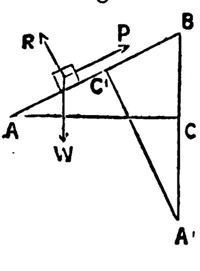


Fig. 2.

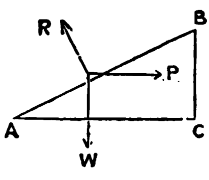
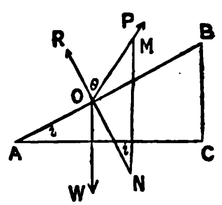


Fig. 3.



279. *If the power acts parallel to the length* take (Fig. 1) $BC'=BC$, draw CA' at right angles to AB and produce BC to A'. Then the triangle BCA' has its sides respectively equal to those of BCA. But the sides of BCA' are parallel respectively to the directions of the forces P, W, R, therefore, (Art. 174),

$$\begin{aligned} P : W : R &:: BC' : A'B : A'C' \\ &:: BC : AB : AC. \end{aligned}$$

That is if the *perpendicular* be taken to represent the power, the *length* represents the weight, and the *base* the pressure on the plane; or

$$P : W : R :: \text{Perpendicular} : \text{Length} : \text{Base}.$$

280. *If the power is parallel to the base*, then (Fig. 2) the sides of the triangle BCA are respectively at right angles to the forces, therefore (Art. 175),

$$\begin{aligned} P : W : R &:: BC : AC : AB \\ &:: \text{Height} : \text{Base} : \text{Length}. \end{aligned}$$

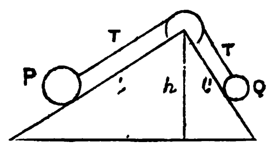
281. *If the Power acts as in Fig.* 3.—Take any point M in the line OP, and draw MN vertically, meeting at N the perpendicular to the plane produced through O. Then (Art. 174),

$$\begin{aligned} P : W : R &:: OM : MN : ON \\ &:: \sin ONM : \sin MON : \sin OMN \\ &:: \sin i \quad : \sin \theta \quad : \sin (\theta - i). \end{aligned}$$

282. All the foregoing relations may be obtained by resolving the forces along the plane and at right angles, and equating the results to zero. (See Art. 211.)

They may also be obtained by the Principle of Work. In Fig. 1, Art. 278, while the power moves through the length of the plane the weight is lifted against gravity through the height of the plane, therefore (Art. 135) $P \times \text{length} = W \times \text{height} \therefore P : W :: \text{height} : \text{length}$. Similarly the other relations may be established.

283. When two bodies support each other on two inclined planes of equal height, they are to one another as the lengths of the planes on which they rest. Let P and Q be the weights, and T the tension of the string. Then (Art. 279),



$$T = P \cdot \frac{h}{l}, \text{ and } T = Q\frac{h}{l'} \quad \therefore \quad P\frac{h}{l} = Q\frac{h}{l'} \quad \therefore \quad P \cdot l' = Q \cdot l \quad \therefore$$
$$P : Q :: l : l'.$$

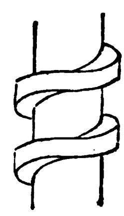
The Wedge.

284. The Wedge is a triangular prism used for cleaving or separating bodies. It is forced between the bodies by blows applied to the back. The wedge may therefore be regarded as a movable double inclined plane, where the power acts parallel to the common base. Any investigation of the action of the instrument as a movable inclined plane would, however, have very little practical value. The forces applied are usually impacts, and the friction is enormous.

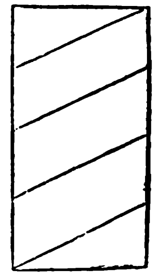
The principle of the wedge is employed in many familiar instruments, such as knives, chisels, nails, and boring and cutting instruments generally.

The Screw.

285. The Screw is a linder surrounded by a uniform spiral projecting thread. The cylinder fits into a cylindrical aperture in a block called the nut, on the inner surface of which is cut a spiral groove corresponding to the thread of the screw. The cylinder, when placed in the block, can only move forwards or backwards by turning on its axis, the thread of the screw working in the groove of the nut. The form of the thread varies in different screws. Its thickness is u ually disregarded in calculations relating to the instrument.



286. The thread may be regarded as a continuous inclined plane wrapped round the cylinder; or it may be considered as a series of inclined planes, each having for its base the circumference of the cylinder, and for its height the distance between two threads. This may be made clear by drawing on paper a rectangle, dividing its length into equal parts, and tracing the parallel lines as in the figure. If the paper be now wrapped round a cylinder whose circumference is equal to the breadth of the rectangle, the parallel lines will be seen to form a continuous spiral round the cylinder.



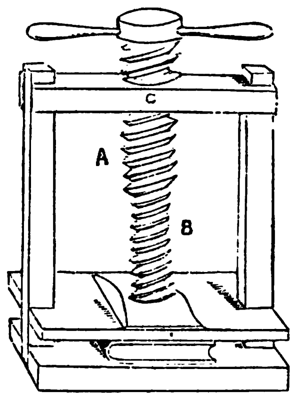
287. The force applied to a screw acts in a plane at right angles to the axis of the cylinder, and is therefore parallel to the bases of the inclined planes surrounding the cylinder. If we suppose the power to act at the circumference of the cylinder, and if we call the power P′, the resistance W, the circumference of the cylinder c, and the distance between two threads d, then (Art. 280), $P' : W :: d : c \therefore P' = W\frac{d}{c}$. But the power is usually applied at the extremity of a lever (Art. 286). Let P be this power and C the circumference which the extremity describes. Then $P : P' :: c : C \therefore P = P'\frac{c}{C}$. Substituting the value of P′ in preceding equation.

Then $P = W\frac{d}{c}\cdot\frac{c}{C} \therefore P{\cdot}C = W{\cdot}d$. That is, *Power × circumference of circle described by power = Resistance × distance between two threads.*

288. The same equation may be obtained more easily and satisfactorily by the Principle of Work. Let the machine be moving uniformly, then the power and resistance are in equilibrium. Let P be the power and C the circumference described in one turn, let W be the resistance and d the distance between two threads. Then in one turn P moves

through a distance C, and W a distance d; therefore, (Art. 135), $P \cdot C = W \cdot d$.

289. The power of the screw may be increased by increasing the length of the lever or diminishing the interval between the threads. It will, however, be readily seen that neither of these methods can be employed beyond certain limits.

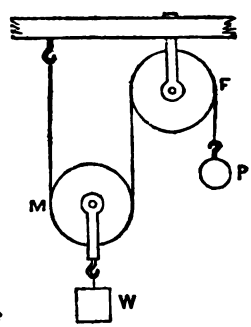


Hunter's Screw or the Differential Screw is constructed on the same principle as the Differential Axle (Art. 274). It consists of two screws A and B of different diameters. A, which works in a nut C, has a hollow screw inside, in which the screw B works. To the lower extremity of B is attached a plate, which by means of guides is allowed only to move up or down. When the handle is turned once round, the screw A moves downwards through a distance equal to the interval between two of its adjacent threads, while the screw B, which works in A, moves upwards through the interval of two of its threads. If the distances were equal, the plate attached to B would remain at the same height, but if the threads of B are nearer to each other than those of A, the distance that the plate moves downwards in one turn is the difference of the intervals of A and B. If d be the interval of the threads of A, and d' of B, and if C be the circumference of the circle described by the power in one turn, then (Art. 135) $P \cdot C = W(d - d')$.

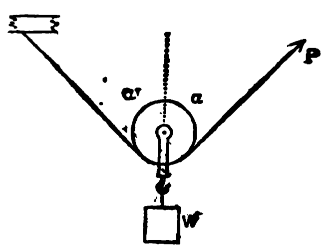
The interval $(d - d')$ may be made as small as we please, and thus the mechanical advantage may be increased without limit. Of course this increase of the mechanical advantage is gained in this as in every other case by a loss in time. If the ratio of W to P is increased, the distance through which W moves compared with that through which P moves is diminished in the same ratio.

The Pulley.

290. The pulley is a small wheel having a groove in its circumference, round which is passed a cord. The wheel turns on an axis supported in a frame called the *block*. When the cord is moved the wheel turns with it, and thus friction is to a great extent avoided. The action of the instrument depends on the tension transmitted by the cord and not on the wheel, whose use is merely to lessen the friction.



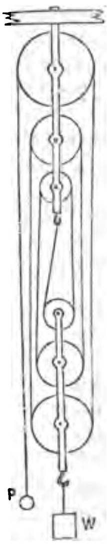
291. With a fixed pulley F no *mechanical advantage* is gained. It merely serves to alter the direction in which the force acts. With a movable pulley a mechanical advantage is gained. If we neglect friction, the tension of the string throughout is the same. In the Fig. this tension is P. If the two parts of the string on each side of the movable pulley be parallel, M and W are supported by two tensions each equal to P, and therefore (Art. 185) $M + W = 2P$. If the weight of M be neglected, $W = 2P$. Therefore the mechanical advantage is 2.



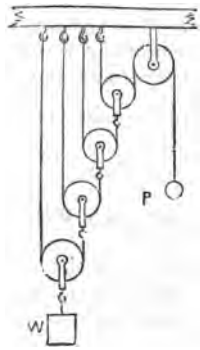
292. If the strings be not parallel the relation between W and P when in equilibrium is found by resolving the tensions vertically. If the angle between the two parts of the string be $2a$, then $W = 2P \cos. a$.

The relations between P and W when in equilibrium may be readily determined in this way for any arrangement of pulleys.

293. *System of Pulleys with a single string.*



Supposing the parts of the string are parallel and neglecting the weights of the pulleys; then, since the tension is the same throughout the same string, each movable pulley is supported by *two* tensions each equal to the power P. Therefore twice the number of movable pulleys multiplied by $P = W$. If $n =$ number of movable pulleys, $W = P \times 2n$.



If the weights of the movable pulleys are to be taken into account, and that the sum of their weights is w, then $W + w = P \times 2n$. So that W will be less than before by w.

294. *Systems of Pulleys with separate strings.*

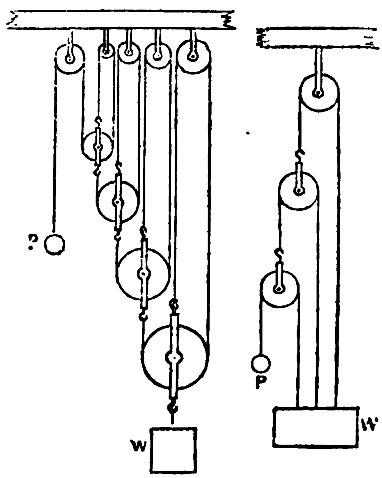
(1.) *Where each pulley hangs by a separate cord attached to a fixed support.*—Let the power be P, and let the weights of the pulleys be neglected. Then, as in Art. 291, the first movable pulley is pulled upwards by two tensions each equal to P, and therefore the tension under the first movable pulley is 2P. The second pulley is pulled upwards by two tensions each 2P, therefore the tension under the second is 4P. Similarly under the third it is 8P. Therefore when there are three movable pulleys $W = 8P$. If n be the number of movable pulleys, the successive tensions under 1st, 2nd, ... nth pulley are $2P$, 2^2P, 2^3P, 2^4P . . . . 2^nP; therefore $W = P.2^n$.

If the weights of the pulleys are to be taken into account, we determine the successive tensions by doubling the tension above each pulley and subtracting the weight of the pulley. A general expression for the relation between W and P can be readily obtained, but the student is recommended to

work each exercise by first principles and not by any formula. Thus if there are 4 movable pulleys each of which weighs 1 lb., and that a power of 10 lbs. is applied, the successive tensions *under* each movable pulley are $2 \times 10 - 1 = 19$; $2 \times 19 - 1 = 37$; $2 \times 37 - 1 = 73$; $2 \times 73 - 1 = 145$. Therefore $W = 145$ lbs.

Similarly we can obtain the power if the weights be given by reversing the above process. If $W = 145$ lbs., then the tensions *above* each pulley are $\frac{145 + 1}{2} = 73$; $\frac{73 + 1}{2} = 37$; $\frac{37 + 1}{2} = 19$; $\frac{19 + 1}{2} = 10$. Therefore $P = 10$ lbs.

295. (2.) *Where each Pulley is supported by three tensions.*



Here if the weights of the pulleys are neglected, the successive tensions under the pulleys are 3P, 9P, 27P, &c. If n be the number of movable pulleys, then the tensions are $P.3$; $P.3^2$; $P.3^3 \dots P.3^n$. Therefore $W = P.3^n$.

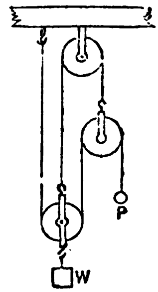
We may take account of the weights of the pulleys precisely as in the last Art. And if W be given, P may be determined by reversing the foregoing process.

296. (3.) *Where each string is attached to the weight.*

Here the weight is equal to the sum of the tensions supporting it. If there be n pulleys and P be the power, these tensions are P, 2P, 4P, 8P, &c. Therefore, $W = P\ (1 + 2 + 2^2 + 2^3 \dots 2^{n-1}) = P\ (2^n - 1)$.

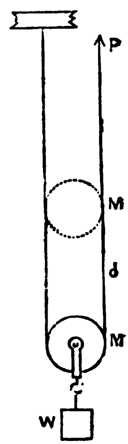
If the weights of the pulleys be each 1 lb., and that there

are three pulleys and a power of 10 lbs., then the tensions supporting the weight are 10 lbs.; $2 \times 10 + 1 = 21$ lbs.; $2 \times 21 + 1 = 43$ lbs. Therefore $W = 10 + 21 + 43 = 74$ lbs.



297. The pressure on the beam or fixed support is in all cases equal to the sum of the tensions of the strings or ties connected with the fixed support.

298. The relation between P and W may be determined in a similar way in any system of pulleys. For instance in the annexed arrangement, by determining the tensions supporting W, we find that, neglecting the weights of the pulleys, $W = 4P$.



299. All of the foregoing relations may be readily obtained by the Principle of Work. If a movable pulley M be raised to the position M′ through any height d, it is evident that P must move through double the distance d in order to keep the string stretched. Hence, if the machine be moving uniformly, $W \cdot d = P \cdot 2d$. Therefore $W = 2P$, as in Art. 291.

Similarly if the Principle of Work be applied to any of the foregoing systems of pulleys, the relations that have already been demonstrated can be easily obtained.

EXAMPLES.

1. A body whose true weight is $9\frac{1}{2}$ lbs. appears to weigh 9 lbs. in one scale of a false balance:—

(*a*). What will it appear to weigh in the other scale?

(*b*). What is the ratio of the lengths of the arms?

(*c*). How may the true weight of a body be found with this balance?

(*a*). The arms are as $9\frac{1}{2} : 9$, or as $19 : 18$, therefore the apparent weight in the other scale $= \frac{9\frac{1}{2} \times 19}{18} = 10\frac{1}{36}$ lbs.

(*b*). $19 : 18$.

(*c*). See Art. 264.

2. A single movable pulley is supported by a cord, one end of which is fastened to a beam, and the other end passed over a fixed pulley attached to the same beam. A man whose weight is 160 lbs. places himself in a basket attached to the movable pulley, and taking hold of the free end of the cord, pulls himself upward. Neglecting friction and the weight of the basket and pulley, and supposing the cords parallel, determine the force he exerts when there is equilibrium.

When in equilibrium his weight is supported by three tensions, each equal to the force with which he pulls. Therefore $\frac{160}{3} = 53\frac{1}{3}$ lbs., the force exerted.

3. A weight of 50 lbs. is supported on an inclined plane by a power of 30 lbs. acting parallel to the length. The height of the plane is 6 feet: what is the length, and what is the pressure upon it? By Art. 279, $30 : 50 :: 6 :$ length $\therefore$ length $= 10$ feet $\therefore$ base $= 8$ ft. And $30 :$ Pressure on plane $: 6 : 8 \therefore$ Pressure $= 40$ lbs.

4. A power of 10 lbs. is applied at the extremity of a lever 5 inches long, which works a screw containing 4 threads to the inch: with what force will the screw be urged?

$$\text{Art. 287, } W = \frac{P \cdot C}{d} = \frac{10 \times 10 \times 3 \cdot 14159}{\frac{1}{4}} = 1256 \cdot 636 \text{ lbs.}$$

5. If the force required to overcome friction on an ordinary road be $\frac{1}{10}$th the load, and that required on a tram-line be $\frac{1}{200}$th of the load, what force will be required in each case to ascend an incline of 1 in 10?

The friction on the incline is supposed to be the same as that on the level road; therefore as the incline rises 1 in 10, $\frac{1}{10}$th of the load in each case is added to the force required on the level road (Art. 279). Therefore—

$\frac{1}{10} + \frac{1}{10} = \frac{1}{5}$th load = force required on the inclined ordinary road; and
$\frac{1}{200} + \frac{1}{10} = \frac{21}{200}$th of load = force required on tram-line ascent.

It will be seen that in each case the incline adds the same fraction of the load; but this fraction bears a much less ratio to the original force in the former case than in the latter. The force required to make the ascent on the ordinary road is in this instance only double the original force; on the tram-line it is twenty-one times as great. Hence the gradients on tram-lines should be much less than those on ordinary roads.

For Examples upon Pulleys see Arts. 294-296.

Exercises.

1. Show that the wheel and axle may be regarded as a system of levers; and from the rule which expresses the relation between the power and weight in the lever deduce the rule for the wheel and axle.

2. With a balance whose arms are unequal the true weight of a body may be found: (*a*) by an experimental method; (*b*) by a method partly experimental and partly theoretical. Explain each mode, and give the demonstration of the latter

3. Two inclined planes whose lengths are 15 feet and 20 feet respectively have a common height. A weight P on one of the planes supports a weight Q on the other by means of a cord which passes over the common vertex of the planes: state and prove the ratio between P and Q.

4. Describe the common steelyard and the Danish steelyard. Show how each is graduated.

5. Two men of the same height bear a cask, whose weight is 1 cwt., suspended from a pole which rests on their shoulders. The cask hangs from a point distant 4 feet from one of the men, and 5 feet from the other: find the weight borne by each.

6. Two weights of 12 lbs. and 18 lbs. are attached to the extremities of a rod 10 feet long: where must a fulcrum be placed in order that the weights may equilibrate each other?

7. To which of the three kinds of levers do the following instruments respectively belong? Assign in each case the reason for your classification: —The oar of a boat; the handle of a pump; a crowbar; a wheelbarrow; a balance; a pair of tongs; a pair of scissors. Point out an example of each of the three kinds of levers in the human body.

8. Sketch a system of pulleys with a single cord and having four movable pulleys. Find the power which will sustain by means of this system a weight of 160 lbs.: (*a*) When the weights of the pulleys are neglected; (*b*) When the weights are taken into account, each pulley weighing 1 lb.

9. Sketch a system of pulleys with three movable pulleys, each having a separate cord, the end of which is attached to a beam. Find the power which will sustain by means of this system a weight of 100 lbs.: (*a*) When the weights of the pulleys are neglected; (*b*) When the weights are taken into account, each pulley weighing 1 lb.

10. In a system of pulleys with one cord, a power of 6 lbs. supports a weight of 48 lbs.: Find the number of pulleys.

11. A weight is placed on an inclined plane whose inclination is i, and is kept at rest by a force which makes an angle θ with the perpendicular to the plane. Investigate the relation between the power, the weight, and the pressure on the plane.

12. In a system of pulleys when there are n cords, each of which is attached to the weight, state and prove the formula which expresses the relation between the power and weight when in equilibrium.

13. On an inclined plane whose length is 10 feet and height 6 feet a weight of 112 lbs. is kept in equilibrium by a power acting parallel to the base. Determine this power and the pressure on the plane.

14. The arms of a false balance are 11 and 12 inches respectively, and the shopkeeper always places the weights in the scales attached to the longer arm. Does he gain or lose by so doing in selling his goods; and how much in every cwt. he sells?

15. With a single movable pulley the power and weight are in equilibrium. What is the relation between them? Give the reasons for your answer.

16. Draw a sketch of any system of pulleys containing three movable

pulleys, each of which has a separate cord, and determine the weight that would be supported with this system by a power of 20 lbs.

17. In the foregoing arrangement if the weight be raised through 1 ft., through what space will the power move?

18. Explain why in a well constructed balance: (*a*) The point of suspension should not coincide with the centre of gravity of the balance; (*b*) The point of suspension should not be below the centre of gravity; (*c*) The fulcrum should be a knife edge.

19. A pair of scales has one arm longer than the other. A body whose true weight is 12 lbs. appears to weigh only 10 lbs. when placed in one of the scales. What will be its apparent weight when placed in the other scale?

20. A bar 12 feet long has a weight of 4 lbs. suspended from one extremity, 10 lbs. from the other extremity, and 6 lbs. from the middle point. Neglecting the weight of the bar, find the point on which it will balance.

21. A body whose true weight is 20 ozs. appears to weigh 24 ozs. in one scale of a false balance. What will be its apparent weight when placed in the other scale?

22. In a steelyard the counterpoise is 1 lb. and the beam and scale pan weigh 2 lbs. The fulcrum is 4 inches from the point of suspension of scale pan, and 1 inch from the C. G. of beam and scale pan, and lies between these points. Find the position of the counterpoise when a weight of 6 lbs. is placed in the scale pan.

23. A straight uniform bar whose weight is 10 lbs. and length 6 feet has a weight of 8 lbs. attached to one extremity, and balances on a fulcrum near this end. What is the position of the fulcrum and what is the pressure upon it?

24. A beam whose length is 10 feet balances on a point 2 feet from the thicker end; but when a weight of 120 lbs. is attached to the other extremity, it balances on a point 2 feet from that end. What is the weight of the beam?

25. If the force required on a railroad to overcome friction and the resistance of the air be 10 lbs. per ton, what force will be required to move a train weighing 100 tons: (*a*) On the level portion of the line; (*b*) On an incline of 1 in 100?

26. If a horse has to exert a force equal to $\frac{1}{10}$th the load to draw a cart along the level road, what force will he have to exert in going up an incline of 1 in 10 with a load of one ton, the friction being the same?

27. Explain why the gradients on railroads should always be much less than those on ordinary roads.

28. A wheel and axle is working uniformly, and the power is observed to descend 10 feet while the weight, which is 1 ton, rises three inches. Find the power.

29. What is the mechanical advantage in a combination of three levers whose arms are as 4 : 1, 7 : 2, and 5 : 2?

30. The arms of a false balance are to each other as 7 to 8, and the

weight is put into one scale as often as the other: what will be the gain or loss per cwt. to the seller?

31. A windlass whose axle is $1\frac{1}{2}$ ft. in circumference is worked by a handle at the end of which a man pushes with a force of 120 lbs.: what weight will he support, the end of the handle describing a circle of 12 feet?

32. A plane rises 7 in 25: what force parallel to the length, and what force parallel to the base will support a weight of a ton?

33. A power of 10 lbs. acting parallel to the length of a plane supports a weight of $16\frac{2}{3}$ lbs.: what is the ratio of the height to the length?

34. Two inclined planes which are respectively 25 and 15 inches in length, and are of the same height, are placed back to back. A weight of 100 lbs. rests on the longer plane, and is connected by a string passing over a pulley at the common vertex with a weight resting on the shorter plane. If there be equilibrium find the weight on the shorter.

35. The circumference of a screw is 15 inches, and the distance between the threads $\frac{1}{4}$ inch: what force at the circumference will overcome a resistance of 100 lbs?

36. What power will be required, acting at the end of an arm 5 feet long, to produce a pressure of half a ton with a screw the threads of which are half an inch apart?

37. What is the mechanical advantage of a screw?

38. A wheel and axle is used to raise a bucket from a well. The radius of the wheel is 15 inches, and while it makes 7 revolutions the bucket, which weighs 30 lbs., rises $5\frac{1}{2}$ feet: what force is required to turn the wheel? (Diameter : circumference :: 7 : 22.)

39. A weight of 112 lbs. attached to one end of a cord which is passed round a fixed pulley, is sustained in equilibrium by a man standing on the ground and pulling the other end of the cord. If the weight of the man be 168 lbs., what is his pressure on the ground?

40. In the foregoing case what is the strain on the beam supporting the fixed pulley?

41. If in question 39 the weight were attached to a system of three movable pulleys, each of which hangs by a separate cord the end of which is attached to a beam, and that the man pulled the cord passing round the first movable pulley, what would be his pressure on the ground?

42. If in the last case each pulley weighed 2 lbs., what would be the man's pressure on the ground?

43. In a system of pulleys with one string when there are 4 movable pulleys, what weight will a force of 50 lbs. support?

44. If in the preceding case the weight of the block with the movable pulleys be 4 lbs., what force would be required to support 1 cwt.?

45. In a system of three movable pulleys each of which is sustained by *three* tensions, what force will be required to sustain a weight of 135 lbs.?

46. What is the mechanical advantage in the preceding case?

47. If the pulleys of question 45 are each 3 lbs. weight, what weight will a power of 20 lbs. sustain?

48. Ten weights, each of 20 lbs., are to be lifted to a height of 8 feet

from the ground. Show how a system of pulleys might be arranged so that, disregarding friction and the weight of the pulleys, all the weights could be lifted together by exerting a force equal to one of them. Show that the distance through which this force would have to act would be the same as when the weights were raised one by one by the same power.

49. In the system of pulleys in which each string is attached to the weight, each pulley weighs $3\frac{1}{2}$ ozs. Find the weight which will be supported by the pulleys alone when there are five movable pulleys.

50. Two levers form a combination. The arms of the first are 12 ft. and 1 ft., and of the second are 25 ft. and 2 ft. A weight of 150 lbs. is attached to the shorter arm of the latter: what power acting at the longer arm of the first lever will equilibrate this weight?

51. Twelve men, each exerting a force of 42 lbs., work a capstan with a length of lever of $7\frac{1}{2}$ feet, the radius of the capstan being $1\frac{1}{4}$ feet: what weight can they support?

52. A lever 8 feet long rests with one end on a fulcrum, and at the other a force P acts vertically upwards. A weight of 40 lbs. is suspended from a point 3 feet from the fulcrum. Find the force P and the pressure on the fulcrum.

53. In rowing a boat if the oar be 12 feet long and the rowlock 3 feet from the part grasped by the hand, what resistance will be equilibrated by a pull of 30 lbs.?

54. In a windlass the thickness of the rope coiled round the axle is 1 inch, and the resistance may be supposed to act along the centre of the cord. The arm of the power is 36 inches, and the radius of the axle is 3 inches: what weight will be supported by a power of 120 lbs.?

55. A screw has 150 threads in 1 ft.: what force acting at the end of an arm 6 feet long will balance a resistance of 1 ton?

56. In a combination of four wheels and axles, the circumference of each wheel is five times that of the axle: what weight will be supported by a power of 3 lbs?

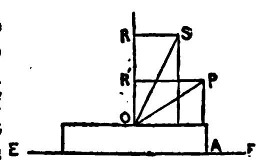
57. With a differential wheel and axle a power of 50 lbs. describes a circumference of 10 feet, and the circumferences of the two parts of the axle are respectively 1 ft. and ·9 ft.: what weight will be sustained?

58. If in the preceding case the weight is raised one inch, through what distance must the power descend?

[*] CHAPTER XVII.

FRICTION.

300. Up to the present in considering the reactions between bodies, their surfaces have been regarded as perfectly smooth. The only reactions possible between such bodies are in directions at right angles to the surfaces in contact. If we press a pencil perpendicularly on a plate of polished glass, the reaction of the surface will counteract the pressure, and the pencil will remain at rest. A very slight obliquity can also be given to the pencil without motion taking place, because the surfaces are not perfectly smooth. If the pencil be pressed on a rougher surface a greater inclination can be given to it before it slides. Thus, when one rough body is caused to move over another, a certain force is required to overcome the roughnesses of the surfaces in contact. The force which opposes the motion of one rough body upon another is called *friction*.



301. *Angle of Friction.*—On the fixed horizontal plane EF let a slab A be placed; then the weight of A is counteracted by the reaction of the fixed plane. Let a force act vertically downwards on A, this is also met by the reaction of the plane. No motion is produced in either case, and no friction is called into action between the surfaces of A and EF, whether these surfaces are rough or smooth. But if a force SO act obliquely on A, this force is equivalent to RO acting vertically, and SR horizontally. The former is counteracted by the reaction of the plane, the latter tends to move A along the plane towards E. This component is opposed by the friction of the surfaces, which acts in the opposite direction, and which may

prevent sliding. If, however, the pressure on A be made more oblique, the horizontal component becomes greater, and may at last become equal to the friction, and then motion is on the point of taking place. Let PO represent the force acting at this degree of obliquity, then the horizontal force PR′ is equal and opposite to the friction, and the angle of obliquity R′OP is called the Angle of Friction. In what follows this angle will be denoted by the symbol ϕ.

302. If in the fig. of the preceding Art. OR′ be the pressure at right angles to the surfaces, and R′P the friction, the line OP will represent in magnitude and direction the resultant of the reactions between the rough surfaces in contact.

303. *Statical Friction.*—Different forces of friction are, therefore, called into exercise between two surfaces by different inclinations of the same pressure, and when the pressure has an obliquity equal to the angle of friction, the greatest amount of friction with that pressure is produced. This maximum value of the force which opposes sliding when motion is on the point of taking place is called the *statical friction.* It is always the same for the same two surfaces, if the pressure remain unaltered, but it varies with the pressure.

304. *Kinetic Friction.*—When one surface slides over another a force of friction opposes the motion. It remains constant for the same surfaces with the same pressure, and is independent of the velocity. This is *sliding friction.* Its amount is always less than the statical friction for the same substances and the same pressure.

When one surface *rolls* on another, as, for instance, when a wheel rolls on a plane, or an axle on its bearing, the friction is found to follow the same laws as sliding friction, but its amount is less than the latter for the same surfaces with the same pressure.

305. *Laws of Friction.*—The following laws have been determined by experiment:—

(1.) *For the same surfaces the friction is proportional to the normal pressure.*

(2.) *For the same pressure the friction is independent of the extent of the surfaces in contact.*

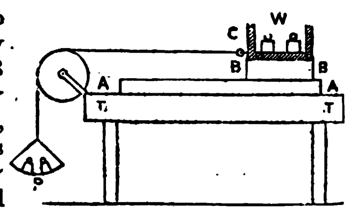
(3.) *The friction between different bodies is different.*

(4.) *The friction increases with the duration of the contact up to a certain time which is different for different bodies, after which it becomes constant.*

These laws apply to both statical and kinetic friction. For the friction of motion the following law in addition has been established:—

(5.) *Kinetic friction is independent of the velocity.*

306. These laws were established by Coulomb by means of the following arrangement:—



TT is a table upon the top of which a block AA of any material may be placed, BB is a block or slab of any substance resting upon AA, and fastened firmly to BB is a box C which can hold any weight W. To BB is attached a cord which passes over a pulley on the edge of the table, and which has attached to its extremity a scale pan in which weights may be placed. Different bodies can be readily placed in the positions of AA and BB; the weights in the box, and the weight of the box and block BB form the pressure, which can be varied at will, and the weight P represents the friction for any pressure.

307. The laws of friction, deduced from experiments with some arrangement such as that of the preceding Art., are true only within certain limits. For instance, in the case of the first law, the pressure must not be so great as to alter the physical state of the bodies. The second law again does not hold if either surfaces be diminished to a physical line or point.

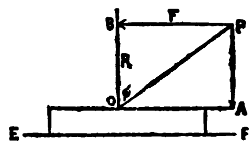
By *normal pressure* is meant the pressure at right angles to the surfaces if they are plane, and if one or both are curved the pressure at right angles to the tangent plane at the point of contact of the two surfaces.

308. *Coefficient of Friction.*—By the first law of friction

the ratio of the friction to the normal pressure is constant for the same two bodies. This ratio is called *the coefficient of friction*. It has been agreed to denote it by the symbol μ. The value of μ with the same surfaces is the same for all pressures, but of course μ is different for different surfaces. If F denote the maximum friction between two bodies, and R the normal pressure, then $\frac{F}{R} = \mu =$ coefficient of friction.

Thus if μ be known for any two bodies, and also the normal pressure R, the friction F is given by the equation, $F = \mu R$.

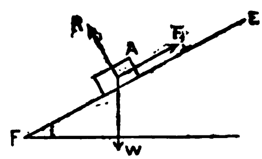
309. *Relation between the Coefficient of Friction and the Angle of Friction.*—The coefficient of friction is equal to the tangent of the angle of friction.



Let PO be a pressure acting at an obliquity equal to the angle of friction; then motion is on the point of taking place, and the angle $POB = \phi$. PO may be resolved into BO, the normal pressure = R, and PB, the friction = F. Hence, $\frac{PB}{BO} = \frac{F}{R} = \tan\phi$. But (Art. 308), $\frac{F}{R} = \mu$; therefore $\mu = \tan\phi$.

Hence if the angle of friction be determined as in Art. 301, the tangent of this angle gives the coefficient of friction; and if the coefficient of friction be found by experiment as in Art. 306, the angle whose tangent is μ is the angle of friction.

310. The angle of friction for two bodies may also be determined as follows:—



A plane surface of one of the bodies A is placed on a plane surface of the other body, and this latter is tilted to such an inclination i with the horizontal plane that A is just on the point of sliding down. The angle i is then the angle of friction. Let the

weight of the body A be W, then A is kept at rest by three forces, the friction $= \mu R$ acting up the plane, R the pressure at right angles to the plane, and W acting vertically. Resolving along the plane and at right angles, then (Art. 211)—

$$F - W \sin i = 0$$
$$R - W \cos i = 0$$
$$\therefore \frac{F}{R} = \tan i.$$

But $\frac{F}{R} = \tan \phi$ (Art. 309) $\therefore i = \phi$; that is the inclination is equal to the angle of friction.

By this means we can readily determine the angle of friction for any two substances, and by taking the tangent of this angle we have the coefficient of friction for the same bodies.

311. The coefficient of friction has been determined in this and other ways for a great number of bodies. The value of μ for statical friction has been found for stone on stone to vary from ·8 to ·6; for wood on wood ·6 to ·4; for metals on metals ·25 to ·15; and for wood on metals ·6 to ·3. The value of μ for any two surfaces is found to be much diminished by smearing the surfaces with grease. As stated in the fourth of the laws of friction the value of μ also depends upon the duration of contact of the surfaces. In the case of metals the maximum value of μ is reached in a few seconds, with woods in one or two minutes, and with wood and metal in a few days.

312. Friction is a resisting force which always opposes motion. Friction, therefore, always helps the weaker force. In the case of statical friction its direction and amount for the state bordering on motion can be determined in the ways that have been explained, and it is this maximum value of the resisting force that is called the friction. If for instance a block whose weight is W rest on a horizontal table and be pulled by a horizontal force P, then, when the block is on the point of moving, the force of friction F acting in the opposite direction, is the maximum friction, and by Art. 308 its value is μW. P in this case $= F = \mu W$.

If W is in motion under the influence of a force P′, then

F the kinetic friction is equal to μW; but μ has a less valuo than in the previous case. The force maintaining motion is in this case P$'-\mu$W, and as this force is measured by its momentum in the unit of time $\therefore$ P$'-\mu$W$=mf$; where m is the mass of the body moved and f the acceleration.

313. *Acceleration on a Rough Inclined Plane.*—Let a body whose weight is W be sliding down a rough inclined plane the inclination of which is i, it is required to find the acceleration. Let $\mu=$ the coefficient of kinetic friction. Resolve W along the plane and at right angles. The component down the plane is W sin i, and at right angles is W cos $i=$ R. As the body is moving down the plane the friction acts up the plane, and is $=\mu$R$=\mu$W cos i. The resultant force down the plane is therefore W sin $i-$ μW cos i.

By Art. 78 the acceleration produced by the force is given by the proportion

$$\text{W} : \text{W} (\sin i - \mu \cos i) :: g : f$$
$$\therefore f = g (\sin i - \mu \cos i)$$

—which is the acceleration down the plane.

Similarly if the body were moving up the plane it may be shown that the acceleration *downwards* is g (sin $i+\mu$ cos i).

Thus problems relating to motion on *rough* inclined planes may be solved by the equations of Art. 14 by writing g (sin $i-\mu$ cos i), or g (sin $i+\mu$ cos i), for g. (See Art. 43.)

Friction in the Simple Machines.

314. In considering the effects of friction in the mechanical powers, we may consider these machines at rest in equilibrium under the forces acting on them, of which friction is one; or we may consider the machines in motion and doing work, and we may proceed to determine the amount of the work which is done against friction. We now proceed to examine the action of some of the simple machines, taking friction into account.

315. *The Lever with Friction.*—A lever may have a horizontal axle attached to it which rests in sockets or

bearings at each end of the axle; or the lever may be pierced with a hole through which passes a cylindrical axle on which the lever rests, and round which it works. We shall consider the latter case.

Let AB be a bar in which a cylindrical hole is pierced, the section of which is represented by the larger circle in the

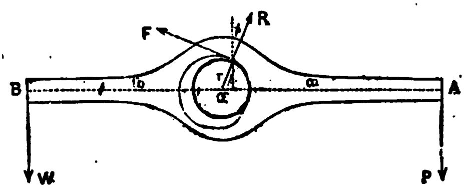


diagram. Let O be the centre of the circle and r its radius, and let a and b be the distances respectively of O from A and B. Let the smaller circle represent the section of the axle on which the lever rests, and let C be the point where the surfaces are in contact. Suppose P is on the point of producing motion in the direction of the hands of a clock. Then the friction $F = \mu R$, where R is the pressure normal to the surfaces in contact; and in this case R makes an angle ϕ = the angle of friction with the vertical through C. For calling this angle α and resolving horizontally, then, since the forces are in equilibrium (Art. 214),

$$R \sin \alpha - F \cos \alpha = 0,$$

But $F = \mu R$, when μ is the coefficient of friction

$$\therefore \sin \alpha - \mu \cos \alpha = 0$$

$$\tan \alpha = \mu \therefore \alpha = \phi \text{ (Art. 309).}$$

Take moments round C, then

$$\frac{P}{W} = \frac{b + r \sin \phi}{a - r \sin \phi}. \text{ But since } \tan \phi = \mu \therefore \frac{\sin \phi}{\cos \phi} = \mu$$

$$\therefore \sin \phi = \mu \cos \phi = \frac{\mu}{\sec \phi} = \frac{\mu}{\sqrt{1 + \mu^2}}$$

$$\therefore \frac{P}{W} = \frac{b + \dfrac{r\mu}{\sqrt{1 + \mu^2}}}{a - \dfrac{r\mu}{\sqrt{1 + \mu^2}}} = \frac{b\sqrt{1 + \mu^2} + \mu r}{a\sqrt{1 + \mu^2} - \mu r}.$$

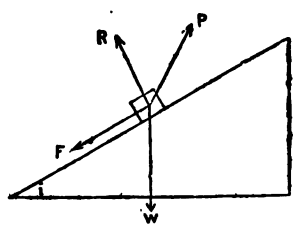
If on the other hand W were on the point of overcoming P and producing motion in the direction opposite to that of the hands of a clock, then we should find in an exactly similar way,

$$\frac{P}{W}=\frac{b-r\sin\phi}{a+r\sin\phi}=\frac{b\sqrt{1+\mu^2}-\mu r}{a\sqrt{1+\mu^2}+\mu r}.$$

Hence the bar will be in equilibrium if the ratio $\frac{P}{W}$ lies between these two extreme values when motion is about taking place in opposite directions.

316. *The Wheel and Axle with Friction.*—The formulae which have been deduced for the lever may be applied to the wheel and axle: a denoting the radius of the wheel, b that of the axle, and r the radius of the shaft or axis on which the axle turns.

317. *The Inclined Plane with Friction.*—Let i be the inclination of the plane, W a body resting on the rough plane and on the point of moving up the plane under the action of a force P which makes an angle θ with the plane. Let R be the pressure at right angles to the plane, then $F=\mu R$ acts down the plane. Resolving along the plane and at right angles, and equating the results to zero (Art. 282),



$$P\cos\theta-\mu R-W\sin i=0$$
$$R+P\sin\theta-W\cos i=0.$$

Eliminating R by multiplying the first equation by μ and adding to the second, then

$$P\cos\theta+\mu P\sin\theta-W\sin i-\mu W\cos i=0$$

$$\therefore\frac{P}{W}=\frac{\sin i+\mu\cos i}{\cos\theta+\mu\sin\theta}. \qquad (1)$$

If W were on the point of moving down the plane, then F

would act up the plane, and we should find in a similar manner,

$$\frac{P}{W}=\frac{\sin i-\mu\cos i}{\cos\theta-\mu\sin\theta}. \qquad (2)$$

318. When the force acts parallel to the length we should find in exactly the same way as the foregoing

$$\frac{P}{W}=\sin i+\mu\cos i \qquad (3)$$

$$\frac{P}{W}=\sin i-\mu\cos i. \qquad (4)$$

It will be seen that these equations may be obtained from (1) and (2) of preceding Art. by putting $\theta=0$ in those equations.

All these results may also be obtained by the Principle of Work, in the way explained in Art. 128.

319. If the body W rest on a horizontal plane and be acted on by a force P inclined at an angle θ to the plane, then in a similar way we should find that when W is on the point of moving,

$$\frac{P}{W}=\frac{\mu}{\cos\theta+\mu\sin\theta}.$$

It will be seen that this result can also be obtained by putting $i=0$ in equation (1), Art. 317.

320. *To determine the most advantageous angle at which a force may be exerted on a body placed on a horizontal plane.*

Let W be the weight of the body placed on a horizontal plane and acted on by a power P at an angle θ to the plane, it is required to determine that value of θ for which the least power will be required to move the body on the plane. From Art. 319,

$P=\dfrac{W\mu}{\cos\theta+\mu\sin\theta}$. Substituting for μ its value $\tan\phi$ (Art. 309),

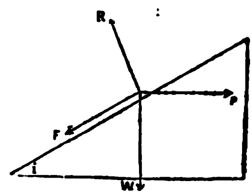
$$P=\frac{W\tan\phi}{\cos\theta+\tan\phi\sin\theta}=\frac{W\sin\phi}{\cos\theta\cos\phi+\sin\theta\sin\phi}=\frac{W\sin\phi}{\cos(\theta-\phi)}$$

Now when P has its least value, $\frac{W \sin \phi}{\cos (\theta - \phi)}$ has its least value and therefore $\cos (\theta - \phi)$ its greatest value. The latter is greatest when $\theta - \phi = 0$ and when consequently $\theta = \phi$. Hence when the power acts at an angle with the plane equal to the angle of friction, the least power is required to move the body on the plane.

In the case of an inclined plane it may be shown in the same way that an angle with the plane equal to the angle of friction is the most advantageous inclination at which a power can act to move a body up the plane.

321. *The Screw with Friction.*—As shown in Art. 286, the screw may be regarded as a series of inclined planes arranged round a cylinder. Each of these planes has for its base the circumference of the cylinder and for height the distance between two threads. If r be the radius of the cylinder, d the distance between the threads, and i the inclination of the thread, then $2\pi r$ = circumference of cylinder = base of each of the planes, and $\frac{d}{2\pi r} = \tan i$.

Let the resistance be W, P′ the power acting at the circumference of the cylinder parallel to the base, and suppose P′ is on the point of overcoming W and the friction μR. Resolving along the plane and at right angles to it, thus—



$$P' \cos i - \mu R - W \sin i = 0$$
$$R - P' \sin i - W \cos i = 0$$
$$\mu R - \mu P' \sin i - \mu W \cos i = 0$$
$$P' \cos i - \mu P' \sin i - W \sin i - \mu W \cos i = 0$$
$$\frac{P'}{W} = \frac{\sin i + \mu \cos i}{\cos i - \mu \sin i} \qquad (1)$$

If the power act at the extremity of an arm R instead of

at the extremity of the radius of the cylinder, then calling this power P,

$$P : P' :: r : R \therefore P' = P\frac{R}{r}.\ \text{Substituting in (1)}$$

$$\frac{P}{W} = \frac{r}{R} \cdot \frac{\sin i + \mu \cos i}{\cos i - \mu \sin i}. \qquad (2)$$

This equation may be put into a form not containing the functions of the angle i. Dividing the terms of the last fraction in (2) by $\cos i$, then

$$\frac{P}{W} = \frac{r}{R} \cdot \frac{\tan i + \mu}{1 - \mu \tan i}.\ \text{But } \tan i = \frac{d}{2\pi r} \therefore$$

$$\frac{P}{W} = \frac{r}{R} \cdot \frac{\frac{d}{2\pi r} + \mu}{1 - \mu \frac{d}{2\pi r}} = \frac{r}{R} \cdot \frac{d + \mu 2\pi r}{2\pi r - \mu d}. \qquad (3)$$

If c = the circumference of the screw, and C the circumference of the circle described by P, the preceding equation may be written

$$\frac{P}{W} = \frac{c}{C} \cdot \frac{d + \mu c}{c - \mu d}. \qquad (4)$$

322. It will be seen from the foregoing investigation, that when a power produces motion in any machine, a part of the force is expended in overcoming the friction of the working parts of the machine. It is consequently an object with mechanicians to diminish friction within certain limits and for this purpose many contrivances are employed. One of these is the substitution of rolling for sliding friction, as in pulleys, the wheels of carriages, &c. But this diminution of friction can be carried only to a certain extent. Friction is essential to the working of all machines. If, for instance, on a railway line the rails and wheels become too smooth the driving wheels will revolve without propelling the train. Friction is an indispensable force in nature. Without it no oblique pressure could be sustained. Without it we could not walk on a horizontal plane, and we could not stand on

an inclined plane. Without friction a ladder could not rest against a wall unless a hole were made in the ground to receive the foot; nails and screws would have no binding power, and structures natural or artificial, such as mountains or houses, could not exist.

EXAMPLES.

1. A weight of 1000 lbs. is placed on a rough inclined plane. The inclination of the plane to the horizon is 45°, and the coefficient of friction is $\frac{1}{\sqrt{3}}$: Find the least force which will draw the body up the plane.

$$\mu=\frac{1}{\sqrt{3}}=\tan 30^\circ \therefore \phi=30^\circ \text{ (Art. 310)},$$

therefore the force required acts at an angle 30° to the plane (Art. 320). Resolving along the plane, and at right angles; thus,

$$P\cos 30^\circ-\mu R-1000\sin 45^\circ=0$$
$$P\sin 30^\circ+R-1000\cos 45^\circ=0$$

Multiplying second equation by μ and adding to the first; thus,

$$P(\cos 30^\circ+\mu\sin 30^\circ)=1000(\sin 45^\circ+\mu\cos 45^\circ).$$

$$\therefore P=1000\frac{\sin 45^\circ+\mu\cos 45^\circ}{\cos 30^\circ+\mu\sin 30^\circ}=1000\frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}\cdot\frac{1}{2}}=996 \text{ lbs. nearly}$$

2. A rectangular block of cast iron rests upon an inclined plane of oak, and is upon the point of slipping down, and also upon the point of overturning. The base is 2 feet square, what is its height, the coefficient of friction being ·62?

Since the body is on the point of sliding, the inclination of the plane = angle of friction $=\phi$ (Art. 310). And since the block is on the point of overturning, the vertical through its centre of gravity falls on the boundary of its base, and makes with the side of the block an angle = angle of inclination of the plane $=\phi$.

Let x = length of block, then $\frac{2}{x}=\tan\phi=\mu=\cdot 62 \therefore x=\frac{2}{\cdot 62}=3\cdot 2$ feet.

3. A uniform beam whose weight is W rests with its lower end on a horizontal plane, and its upper against a vertical wall. If μ be the coefficient of friction for the beam and the plane, and μ' for the beam and the wall, determine the inclination of the beam to the horizontal plane when it is just upon the point of sliding, and the pressures upon the plane and the wall.

Let θ = the inclination required, R the normal reaction of the horizontal plane, R′ that of the wall, and let l be the length of the beam. The forces acting on the beam when it is on the point of sliding are the weight W acting at the middle point, the reactions R and R′ at the extremities, and the frictions μR and $\mu' R'$ acting opposite to the directions in which the ends are on the point of moving.

Resolve vertically and horizontally and take moments round the lower end (Art 214); thus,

$$R + \mu' R' - W = 0 \qquad (1)$$

$$\mu R - R' = 0 \qquad (2)$$

$$W\frac{l}{2}\cos\theta - R' l \sin\theta - \mu' R' l \cos\theta = 0 \qquad (3)$$

From equations (1) and (2) $R = \frac{W}{1+\mu\mu'}$, and $R' = \frac{\mu W}{1+\mu\mu'}$.

Substituting in (3) and dividing by $l\cos\theta$, we obtain—

$$\tan\theta = \frac{1-\mu\mu'}{2\mu}.$$

EXERCISES.

1. The inclinations of two rough inclined planes are respectively 30° and 45°, and a body resting on each is just on the point of sliding down. What is the coefficient of friction in each case?

2. At what inclination to the horizon must a wooden way be laid in order that a block of iron may just slide down by its own weight, the coefficient of friction being ·625?

3. Find (1) What force is necessary to sustain, and (2) What force is just insufficient to push a weight of 250 lbs. up a plane inclined at an angle of 45° to the horizon, the direction of the force making an angle of 30° with the plane, and the coefficient of friction being $\frac{1}{3}$.

4. A uniform ladder is placed between a rough horizontal plane and a rough vertical wall at an angle of 45°, the coefficient of friction between the ladder and the ground being $\frac{2}{3}$; a man whose weight is half that of the ladder ascends. Find what the coefficient of friction must be between the ladder and the wall that when the man reaches the top of the ladder it may just begin to slide.

5. A body weighing 54 lbs. is just set in motion on a rough horizontal plane by a horizontal force of 9 lbs. If the force be withdrawn and the plane tilted up, at what inclination of the plane to the horizon will the body begin to slide?

6. A rough plane is inclined at an angle of 30° to the horizon. A weight W is placed on it, and it is found that a force $\frac{3}{4}$W acting parallel to the plane will just move the weight up the plane. Find the coefficient of friction.

7. A uniform ladder 10 feet long rests with one end against a smooth vertical wall, and the other on the ground, the coefficient of friction being

7. Find how high on the ladder a man whose weight is three times that of the ladder may ascend before it begins to slide, the foot of the ladder being 6 feet from the wall.

8. What force acting parallel to the plane will draw a weight of 100 lbs. along a rough horizontal plane, the friction being such that if the plane were tilted to an inclination of 30°, the weight would be just on the point of sliding?

9. In the preceding case what is the most advantageous angle at which the force can act, and what force acting at this angle will be just insufficient to draw the body along the horizontal plane?

10. A force P acting at an angle θ to a plane whose inclination is i sustains a weight W upon the plane. If the coefficient of friction be μ, determine between what limits P must lie in order that there may be equilibrium.

11. A ladder whose C. G. is at its middle point, rests with one end on a horizontal plane and the other on a vertical wall, to which it is inclined at an angle of 45°. The coefficient of friction for the surfaces at the lower end is $\frac{1}{2}$, and at the upper end is $\frac{1}{3}$. A man whose weight is half that of the ladder ascends it: how far will he go before the ladder begins to slide?

12. A body whose weight is 50 lbs. is just kept by friction from sliding down a rough plane whose inclination is 30°: what is the force of friction and the pressure at right angles to the plane?

13. In the preceding case find the magnitude and direction of the resultant reaction of the rough plane on the body.

14. A body whose weight is 100 lbs. is just maintained at rest on a rough inclined plane by the friction. The height of the plane is 5 feet, and its length 25 feet: what is the force of friction?

15. In the preceding case determine the force acting parallel to the plane which is necessary to draw the body up the plane.

16. A power of 30 lbs. just supports a weignt of 50 lbs. on a smooth inclined plane. If the plane were rough and the coefficient of friction $= \frac{1}{12}$, what power, acting along the plane, would draw the weight up the plane.

17. A lever whose arms are 10 feet and 2 feet, works on an axis passing through a circular hole in the lever, the radius of which is 1 inch. If the coefficient of friction be $\frac{1}{2}$, what power, acting at the end of the longer arm, will just be on the point of moving 1000 lbs. attached to the end of the shorter arm?

18. In the preceding case, if there were no friction, what power would be required?

19. A parallelopiped of wood rests on an inclined plane of wood, and is just on the point both of overturning and of sliding down the plane. Its base is 2 feet square: what is its height, the coefficient of friction being $\frac{1}{2}$?

20. The arms of a lever are 10 feet and 2 feet; an axle projecting from the lever rests at each extremity in a socket which has a radius of 6 inches: what power will raise a weight of 100 lbs., the coefficient of friction being $\frac{1}{2}$?

21. What power would be reouired if there were no friction?

22. If the arms of a lever are 12 feet and 4 feet, and the radius of the socket 6 inches, what weight will a power of 20 lbs. raise, the coefficient of friction being $\frac{1}{3}$?

23. A screw is worked by a lever the extremity of which describes a circle of 12 feet. The circumference of the screw is 6 inches, and the interval between the threads is 1 inch. If the coefficient of friction be $\frac{1}{4}$, find the pressure which is exerted by a force of 20 lbs. acting at the end of the lever.

24. If there were no friction in the preceding case, what pressure would be exerted?

25. A body slides from rest down a rough plane whose inclination is 60°. If the coefficient of friction be $\frac{1}{4}$, find the space described in 3 seconds, and the velocity with which the body is then moving.

26. A body just rests on a rough inclined plane when its inclination is 30°: find the velocity which the body will acquire in sliding down this plane for 2 seconds, when it is inclined at an angle of 60°. If the body be projected up the rough plane whose inclination is 30°, with a velocity of 100 feet per second, how far will it rise before it comes to rest?

27. If in the last case the plane were smooth, how far would the body ascend?

28. A body is thrown up an inclined plane whose inclination is i, with a velocity u: find how far it will ascend before it comes to rest (1) when the plane is smooth, (2) when it is rough, and the coefficient of friction $= \mu$.

HINTS FOR THE SOLUTION OF THE EXERCISES.

NOTE.—*The Student should not consult these hints until he has first attempted the solution himself. The problems may be solved in different ways.*

CHAP. II., p. 13. (1) Employ equations of Art. 12, as in Exam. 1. (2) Art. 12; Use equation connecting velocity and time, and solve for t. (3) Art. 12; equation connecting space and time. (4) Art. 11, since there is an initial velocity; (a) equation connecting velocity and time; (b) do. space and time; use + sign. (5) Do.; use − sign. (6) Art. 11; equation connecting velocity and space; when the body comes to rest $v=0$. (7) Art. 11; equation for velocity and time; $v=0$ when body comes to rest. (8) If f be the acceleration, then as in Exam. 2, $132=\frac{1}{2}f(6^2-5^2)$, from which f is found; then use equation 2, Art. 12. (9) Art. 12; equation for space and time; employ units of yards per minute, or reduce to feet per second per second. (10) See Exam. 4. (11) Do., and Exam. 2. (12) Art. 11; equation for velocity and time; $v=0$ when body comes to rest. (13) Do.; equation connecting velocity and space; put $v=0$. (14) Art. 14; equations for initial velocity connecting velocity and time, and space and time; use units of metres per second. (15) Art. 14; equation for initial velocity connecting velocity and space; put $v=0$. (16) Art. 14; equation connecting velocity and time without initial velocity. (17) See Exam. 4. (18) Exam. 2. (19) Art. 14; equation connecting space and time; $s=48$. (20) Art. 11; space and time; + sign. (21) Art. 12: find f from equation for space and time, then find v from equation for velocity and time. (22) Find from equation 2, Art. 14, the times of describing 512 feet and 256 feet respectively, and take the difference. Or, find the velocity after describing the first 256 feet, and then employ equation 5 of Art. 14. (23) Let $t=n^{\circ}$ of secs., then $(t+1)=$ whole time, then as in Exam. 2, $\frac{1}{2}g\{(t+1)^2-t^2\}=176$, solve for t. (24) Art. 14; space and time; solve for t. (25) Find velocity at point B from equation $s=ut+\frac{1}{2}gt^2$, putting $s=144$; then find space through which a body must fall to acquire this velocity by equation 3, Art. 14. (26) See Art. 15. (27) Exam. 7. (28) Express given velocity in yards per minute, and use equation $v=ft$. (29) 352 yards per minute gained per minute $=\frac{352\times 3}{60\times 60}$ feet per second gained per second; Arts. 17 and 18. (30) Arts. 17 and 18. (31) 32 feet per second per second $=\frac{32}{3}\times 60\times 60$ yards per minute per minute.

CHAP. III., p. 25. (1) By Arts. 21, 28, the body will not remain at rest whether the velocities be in the same or in different directions. (2) Art. 21. (3) Art. 26 or 27. (4) Find velocity by Art. 21, and then use equation of Art. 6. (5) Exam. 3. (6) Resolve vertically. (7) Exam. 1. (8) Art. 21. (9) Difference of rates = twice rate of stream. (10) to (15) Art. 26 or 27. (16) and (17) Reduce to same units and compare. (18) Resolve all the velocities in the direction of any one, and apply Art. 21. (19) Employ Art. 32 (*c*). (20) Art. 26 or 27. (21) The three velocities are in directions at right angles to each other, therefore employ Art. 30. (22) Find resultant velocity by Art. 27; then use equation, $s = ut$. (23) Art. 30 and equation $s = \frac{1}{2}ft^2$. (24) See Ex. 22. (25) Resolve horizontally. (26) See Ex. 22. (27) Do. (28) Arts. 27 and 33. (29) Do., and equation, $s = \frac{1}{2}ft^2$. (30) Art. 32 (*c*), and equation, $s = ut$.

CHAP. IV., p. 37. (1) Equations 1 and 2 Art. 47. (2) Equation 2 Art. 47. (3) Put t = time; find space traversed by each, using equation with $-$ sign for 1st, and with $+$ sign for 2nd; sum of spaces = 128. (4) Same way as last; for 2nd stone, use equation $s = \frac{1}{2}gt^2$. (5) Exam. 1 and Art. 47. (6) Art. 47. (7) Equation 2, Art. 47; solve for t. (8) Equation 3, Art. 47; by Art. 47 the velocity is the same in both cases. (9) to (17) These are solved by the equations of Art. 14, writing $g \sin i$ for g. (9) Equation 1. (10) Equation 2. (11) Equation 6 with $-$ sign, put $v = 0$, and solve for s. (12) Equation 2, solve for t. (13) Equation 4; for (*b*) put $v = 0$. (14) Equation 6, put $v = 0$. (15) Equations 4 and 5 with $+$ sign. (16) Equation 2. (17) Equation 5 with $+$ sign. (18) Let t = time of 1st, then $(t-2)$ = time of 2nd; find by equation 2, Art. 14, the space described by each; the difference of these spaces = 192. (19) Method of Exam. 3, but use equations with $+$ sign. (20) Exam. 3. (21) Equation 2 Art. 14, using $g \sin i$ for g; see Art. 44. (22) Do. (23) Find velocity in each case in 5 secs. and compound by Art. 26 or 27. (24) Do. (25) Do.; the angle between the directions is 120°. (26) Do., do. (27) Exam. 6. (28) (*a*) To the height obtained in 27 add space traversed with uniform velocity by balloon. (*b*) To the same height add the height reached by the stone found as in Exam. 1. (29) Exam 6. (30) The vertical space is the same as if there were no horizontal velocity, the horizontal space is the same as if there were no vertical velocity. (31) Art. 36. (32) Ans. to Exercise 30. (33) Do. (34) Do. (35) Equation 2, Art. 47, with s negative. (36) Find velocity in 2 secs. and proceed as in last exercise. (37) Do. (38) Let t = time, and u = velocity of projection; the whole height $= \frac{u^2}{2g}$ (see Exam. 1); find spaces described by each body by equations 2 and 5 of Art. 14; put sum $= \frac{u^2}{2g}$ from which t is found $= \frac{u}{2g}$ space described in this time $= \frac{3}{4}$ the whole space; find time taken to fall through one-fourth the whole height. (39) Equation 2, Art. 14, substituting value of g. (40) Exercise 30. (41) Exam. 4. (42) Let u = velocity of the balloon; $4{\cdot}5\,u$ = height when stone is dropped; substitute

for s in equation 2 of Art. 47 and make s negative; solve for u. (43) Find velocity, by equation 3 Art. 47, then use equation 2 Art. 47. (44) Art. 47. (45) Do. (46) Do. (47) Exam. 6. (48) Arts. 17, 18. (49) Do.; 32 feet per second gained per second $=\frac{12}{11}\times 300\times 300$ units of length gained in the unit of time.

CHAP. V., p. 50. (1) Arts, 60, 61, and 64. Horizontal velocity $=50+$ resolved velocity of projection; the height and time of flight depend on vertical velocity only. (2) Art. 66. (3) Do. (4) Vertical and horizontal velocities are independent of one another. (5) Velocity is horizontal at the middle point of range. (6) See hint to Exercise 4. (7) Do. (8) Exam. 1. (9) Find time of flight; then use Art. 47 to find velocity of projection. (10) Art. 64. (11) Do. (12) Hint to Exer. 4. (13) Find time of reaching greatest height; again, find time of moving from vertex to end of latus rectum $=$ time of falling to focus $=$ time of describing $\frac{u^2\cos^2\alpha}{2g}$; sum of times $=$ time required. (14) Art. 57. (15) Art. 64. (16) Do., solve for u. (17) Greatest height $=64$ feet.

CHAP. VI., p. 64. (1) Exam. 1. (2) Exam. 2. (3) Find acceleration, and use equation 3, Art. 12. (4) Do. and equation 1, do. (5) Find acceleration Art. 90, and use equation 2, Art. 12. (6) Determine velocity, and proceed as in Exam. 2. (7) Find acceleration; find force required to produce this acceleration by (1), Art. 90. (8) Determine unit of mass; measure is inversely as unit; see Arts. 80–82. (9) Exam. 1., or Art. 90. (10) Do. (11) See hint to 7. (12) Do. (13) Exam. 6, p. 13. (14) Find acceleration by Art. 90; use equation 2, of Art. 12. (15) Find acceleration from (2) Art. 12; then use equation in (2) Art. 90, and solve for P; from this value subtract 50. (16) Same equation as in last. (17) Do. (18) Find force, Art. 90, that would produce the acceleration if there were no friction; and add 2 lbs. (19) Find acceleration from equation 2, Art. 12; put $x=$ required weight and use equation (2) of Art. 90. (20) Find acceleration, Art. 90; and proceed as in Exam. 2, p. 12. (21) Find acceleration, Art. 90 (4); then use equation 1, Art. 12. (22) Do., and equation 2, Art. 12. (23) Find acceleration from equation 2, Art. 12; find velocity from equation 1, Art. 12; find space from equation 2, Art 12. (24) Forces are proportional to the accelerations when the mass is constant; the acceleration corresponding to the weight is g. (25) Find acceleration as in Art. 90; the force producing motion is $(5-6\times\frac{1}{2})$, and the mass moved is $(5+6)$; then use equation 3, Art. 12. (26) Find acceleration as in Art. 90; the force producing motion is $\left(300-1120\times\frac{20}{200}\right)$; then use equation 2 of Art. 12, and solve for t. (27) Find acceleration; use equation 3 of Art. 11 with $-$ sign (28) Equation of Art. 89, solve for R. (29) Find acceleration from Art. 90; find velocity from (1) Art. 12; find time from equation of Art. 89. (30) See last. (31) Do. (32) Art. 86. (33) Art. 77. (34) The unit force is that which would produce in 1 lb. a

velocity of 1 ft. per 1". Number of units in $1\frac{1}{2}$ lbs. $= 1\frac{1}{2} \div \frac{1}{32}$ (35) Force of 100 lbs. would produce in mass of 100 lbs. an acceleration of 32 ft. per 1" $\therefore$ would produce in mass 100×16 an acceleration of 2 ft. per 1" $\therefore$ in mass $100 \times 16 \times (\frac{1}{4})^2$ an acceleration of 2 ft. in $\frac{1}{4}$". Or: 2 ft. per $\frac{1}{4}$" per $\frac{1}{4}$" $= 32$ ft. per 1" per 1"; and a force of 100 lbs. will produce this acceleration in a mass of 100 lbs. (36) Force of 2240 lbs. would produce in mass of 2240 lbs. an acceleration of 32 ft. per 1" $\therefore$ a force of $2240 \div \frac{32}{3}$ would produce in same mass an acceleration of 1 yard per 1" $\therefore$ a force of $2240 \div \frac{32}{3} \div 60^2$ would produce an acceleration of 1 yard per minute.

Chap. VII., p. 74. (1) Equation of Art. 97; put $P = 10$, and solve for Q. (2) Art. 98. (3) Do. (4) Momenta are equal and opposite. (5) Momentum gained by one is lost by the other. (6) Art. 100 and Exam. 2. (7) Arts. 97 and 85 (8) Tension to produce uniform velocity $= \frac{1}{8} \times 8$ tons; force to give acceleration of 3 ft. per sec. $= 8 \times \frac{3}{32}$ tons; sum of both $=$ required tension. (9) Find acceleration, Art. 90; find velocity, Art. 12; find time, Art. 89. (10) Exam. 2. (11) Do. (12) Do. (13) Do. (14) Hint to Ex. 8. (15) Art. 99, $T = \frac{108 \times \frac{5}{8}}{21}$. (16) Art. 99. (17) The sine $= 1$ when the plane is vertical, and $= 0$ when horizontal. (18) Art. 98. (19) Proceed as in Art. 99. (20) Do. (21) When one plane is vertical the sine of its inclination $= 1$.

Chap. VIII., p. 84. (1) Art. 116. (2) Do.; $u' = 0$. (3) Do. do. (4) Exam. 1. (5) Art. 115. (6) Do.: make u' negative. (7) Do. do. (8) Exam. 2. (9) Art. 115; $e = 1$. (10) From Exam. 3, the coefficient varies as square root of height of rebound. (11) Art. 115. (12) Velocity of each ball after being struck $=$ that of striking ball. (13) Art. 117; $e = 1$, $m = m'$, $u' = 0$. (14) Do. (15) Exam. 3. (16) Do.; the height on first rebound being e^2h, that on second $= e^4h$, on third $= e^6h$, &c. (17) Art. 117. (18) Let i and r be the angles of incidence and reflection; $\tan r = \frac{1}{e} \tan i$, and $i + r = 90^\circ \therefore \tan(90^\circ - i) = 3 \tan i \therefore \cot i = 3 \tan i \therefore \tan i = \frac{1}{\sqrt{3}} = \tan 30^\circ$. (19) $e = \frac{1}{2}$; put $u =$ velocity of projection; then velocity before and after striking the ceiling is known from Arts. 14 and 112; similarly, velocities before and after striking the floor; the greatest height reached with this latter velocity is given by Art. 14; put this $= 12$, and solve for u. (20) Art. 115; $m = 3m'$, $e = 1$, u' negative; solve for v.

Chap. IX., p. 96. (1) Art. 137. (2) Exam 2. (3) $\frac{660 \times 2240 \times 100}{30 \times 33000}$ $=$ No. of minutes. (4) Art 137. (5) $\frac{60 \times 33000 \times 10 \times 60}{50 \times 6 \times 10} =$ No. of galls. (6) Exam. 1. (7) Exam. 4. (8) Find cubic content, multiply by $62\frac{1}{2}$ and by 4. (9) Find kinetic energy. (10) Exam. 1. (11) Find units

of work per minute, and express as fraction of 33,000. (12) Exam. 2. (13) Find work actually done per hour; find theoretical work per hour; divide former by latter. (14) The lead ball will possess greater energy than the cork if both bodies be projected with the same velocity; apply Art. 127. (15) Determine (*a*) the acceleration, (*b*) the velocity in 3 seconds, (*c*) apply Art. 127 to find resistances. (16) Use equation of Art. 127, and solve for resistance. (17) Find (*a*) the acceleration, (*b*) velocity in 3 secs., (*c*) apply equation of Art. 89. (18) Find (*a*) velocity in 10 secs., (*b*) apply equation of Art. 89 to find time, (*c*) equation of Art. 127 to find space. (19) Exam. 5. (20) Find (*a*) acceleration, (*b*) velocity in 9 secs., (*c*) equation of Art. 127. (21) Arts. 125, 131. (22) Art. 137. (23) Apply equations of Arts. 89 and 127. (24) Do. (25) Art. 127. (26) $\frac{1200 \times 15 \times 5280}{60 \times 33000} = \text{H. P.}$ (27) Add to resistance $\frac{1}{80}$ of weight. (28) Equation of Art. 127. (29) Do. (30) Resistance $= (1400 + 4480)$ lbs. (31) Exam. 6. (32) (*a*) equation of Art. 89, (*b*) $v = g$ sine $i\ t$. (33) Art. 137. (34) Art. 125. (35) (*a*) 500×500, (*b*) $500 \times 500 \times 980$. (36) Kilogramme $=$ 1000 grammes; metre $=$ 100 centimetres; use equation of Art. 125. (37) Let v = velocity, then $120 \times 10 \times v = 30 \times 550$.

CHAP. X., p. 104. (1) $f = \frac{v^2}{r}$ where $f = g$; solve for v. (2) Equation 2, Art. 140. (3) Equation 4, Art. 144. (4) Equation 2, Art. 140; solve for v. (5) Equation 2, Art. 140. (6) Equation 4, Art. 144. (7) Art. 140. (8) Art. 142. (9) Art. 148. (10) Art. 147. (11) Equation 2, Art. 140. (12) Do. (13) Do. See Exam. 1.

CHAP. XI., p. 115. (1) Equation of Art. 154; solve for l. (2) Art. 157. (3) Do. (4) Do. (5) Art. 163. (6) The lengths are proportional to the accelerations due to gravity. (7) Art. 157. (8) See Ex. 6. (9) $32.09 : 32.19 :: x^2 : 86400^2$.

CHAP. XII., p. 125. (1) $R^2 = P^2 + Q^2$ $\therefore$ Divide 15^2 in the ratio of 3^2 to 4^2. (2) Arts. 167 and 174. (3) Art. 167. (4) Use Art. 174. (5) Do. (6) Equation $R^2 = P^2 + Q^2$. (7) Art. 171. (8) Do., or resolve forces in the line bisecting the angle. (9) Resolve horizontally. (10) Angle between forces $= 90°$. (11) Exam. 5. (12) Art. 171. (13) Do. (14) Do.; put $R = Q$. (15) Exam. 4. (16) Do. (17) Equation of Art. 171; solve for cos α. (18) Do.; $\alpha = 60°$. (19) Ex. 2. (20) Art. 177. (21), Exam. 6. (22) Art. 171. (23) Do. (24) Do. (25) Exam. 5. (26) Art. 175. (27) Exam. 5. (28) Resolve horizontally and vertically. (29) Exam. 5. (30) Art. 171. (31) Do. (32) Do. (33) Apply in each case equation of Art. 171. (34) Do. (35) Do.; or, resolve in direction bisecting the angle. (36) Resolve along each diagonal, and compound resultants. (37) Exam. 6. (38) Art. 169. (39) Art. 178. (40) Art. 181 (*c*). (41) Resolve horizontally. (42) With a long rope the direction of the pull can approximate more nearly to the direction of motion. (43) Apply Triangle of Forces.

CHAP. XIII., p. 134. (1) Art. 185. (2) Do. (3) Exam. 1. (4) Do. (5) Art. 189. (6) Art. 194. (7) Let P=each of the forces; the resultant of P and P along BC and AC, is by Art. 171, $P\sqrt{2}$, and is parallel to AB; then apply Art. 186 to find point of application of resultant of P along AB and $P\sqrt{2}$ parallel to AB. (8) Let P, Q, S be the forces acting at A, B, C respectively; then as in Exam. 2 the resultant of P and Q acts at D on CO produced $\therefore$ the resultant of the three forces acts on a point on CD. Similarly it may be shown that it acts on a point on AO produced $\therefore$ it acts at O. (9) Art. 194. (10) Do.

CHAP. XIV., p. 146. (1) One of the forces must be equal and opposite to the resultant of the other two. (2) Art. 208. (3) Art. 212. (4) Take moments round unoccupied angle. (5) Exam. 1. (6) Do. (7) Do. (8) It can be easily shown that the sum of the moments of the forces round any point in the plane is equal to double the area of the polygon; and therefore, Art. 207, the forces are equivalent to a couple. (9) Resolve vertically and horizontally and take moments round A; see Exam. 3. (10) The greater the distance of the bundle from the shoulder the greater the moment, and the greater must therefore be the pull at the other end. The pressure on shoulder = sum of forces at extremities of stick. The reaction at the shoulder is always equal to the pressure on it. The quantity of matter remains unaltered and $\therefore$ the force of the earth's attraction is always the same. (11) Let the weight of the rod be w, acting at the middle point, and let P = pressure on the peg at the extremity; take moments round the other peg $\therefore P = \frac{1}{2}w$; pressure on peg round which moments are taken = sum of pressures. (12) Let x = distance from one peg; take moments round that peg $\therefore 20 \times 6 - 8 \times 12 = 24x \therefore x = 1$ inch from peg or 5 from end. (13) Exam. 3, and Exer. 9. (14) Do. (15) Let x = weight; take moments round C. (16) Resolve vertically and horizontally and take moments round B. (18) Take moments round lower end.

CHAP. XV., p. 165. (1) Art. 226. (2) Art. 230; Answer to second part of question depends on theory of inclined plane, see machines. (3) Art. 226. (4) Art. 240. (5) Arts. 238, 231. (6) Art. 233. (7) Art. 228. (8) Legs are at angular points of equilateral triangle; draw perpendicular from angular point to one side and produce to circumference; this point and the centre of table are equidistant from side. (9) Let A B C be the angular points; resolve 90 lbs. acting at C.G. into 30 lbs. at A, and 60 lbs. at the middle point of BC; then resolve latter force into two components at B and C respectively. (10) Exam. 5. (11) The C.G. is to be equidistant from the props. (12) Exam. 2. (13) Let x = length; find weight which acts at middle point; take moments round fulcrum. (14) Proceed as in Exam. 3. (15) Weights of books act at middle points; resultant passes through edge of table when books are on the point of falling. (16) Take moments as in Exam. 3, round middle point of diagonal, or apply Art. 248. (17) Exam. 3. (18) Do. (19) Ex. 9. (20) Bisect arms, join points of bisection, bisect this line and find distance of middle point from middle point of bent

rod. (21) Take moments round the sides as in Exam. 6. (22) Take moments round common side. (23) Art. 241. (24) Exam. 2. (25) Take moments round centre as in Exam. 3. (26) Do. (27) Do. (29) Take moments round common plane. (30) Exam. 5. (31) Do. (32) Exam. 3. (33) Do. (34) Resolve vertically and horizontally and take moments round lower end. (35) Exam. 5.

Chap. XVI., p. 190. (1) Art. 272. (2) Art. 264. (3) Art. 283. (4) Arts. 266, 269. (5) Resolve into two components; or, take moments round each extremity, (6) Take moments round fulcrum; or, use Art. 185. (7) Art. 254. (8) Art. 293. (9) Art. 294. (10) Art. 293; solve for n. (11) Art. 281. (12) Art. 296. (13) Art. 280. (14) Art. 255. (15) Art. 291. (16) Art. 294. (17) Art. 135. (18) Arts. 260–262. (19) Exam. 1. (20) Let $x=$ distance of fulcrum from middle point; take moments round fulcrum. (21) Exam. 1. (22) Take moments round fulcrum. (23) Ex. 20; weight acts at middle point. (24) Weight acts at point upon which it first balances: take moments round second fulcrum. (25) (a) Force to overcome friction and resistance of air (b) former force, plus force equal $\frac{1}{100}$ of weight. (26) See last Ex., and Exam. 5. (27) Do. (28) Art. 272. (29) Art. 258. (30) Suppose he sells a pound weight each time; then (1) he sells $1\frac{1}{7}$ lb. for 1 lb.; (2) he sells $\frac{7}{8}$ lb. for 1 lb.; he therefore loses $\frac{1}{7}$ lb. and gains $\frac{1}{8}$ lb., by the two sales, therefore he loses $\frac{1}{56}$ lb. on every 2 lbs. he sells. (31) Art. 272. (32) Art. 279. (33) Do. (34) Art. 283. (35) Art. 287. (36) Do. (37) Do. (38) Art. 272. (39) Pressure is lessened by amount of pull. (40) Art. 297. (41) Ex. 39, and Art. 294. (42) Do., do. (43) Art. 293. (44) Do. (45) Art. 295. (46) W : P. (47) Art. 295. (48) Arts. 293 and 299. (49) Proceed as in Art. 296, P being $=0$. (50) Art. 258. (51) Art. 273. (52) Art. 255. (53) Do. (54) Art. 272; radius of axle is increased by $\frac{1}{2}$ inch. (55) Art. 287. (56) Art. 276. (57) Art. 274. (58) Art. 135.

Chap. XVII., p. 207. (1) Arts. 309 and 310. (2) Do. (3) In each case resolve along the plane and at right angles, and proceed as in Exam. 1. In the first case the friction acts up the plane, and in the second down. (4) Exam. 3. Resolve vertically and horizontally and take moments round the lower end. If $d=$distance between foot of ladder and wall, the equation for the moments is $W\cdot\frac{1}{2}d+\frac{1}{2}W\cdot d-R'd-\mu'R'd=0$. Proceed as in Exam. 3. (5) Find μ by Art. 308, and ϕ by Art. 309. (6) Friction acts down the plane. Resolve along the plane and at right angles. (7) Exam. 3. If $\theta=$the inclination of the ladder to the ground, $\sin\theta=\frac{4}{5}$ and $\cos\theta=\frac{3}{5}$. Let $x=$distance ascended; resolve vertically and horizontally, and take moments round lower end. The equation for the moments ($W\cdot 5\cos\theta + 3W\cdot x\cos\theta - R^1\cdot 10\sin\theta=0$) becomes $W\times 3+3W\times x\times\frac{3}{5}-R'\times 8=0$. (8) The coefficient of friction $=\tan 30^\circ=\frac{1}{\sqrt{3}}$; $F=\mu R=\mu W$. (9) Art. 320. (10) Art. 317. (11) Exam. 3, and Ex. 7. (12) Resolve along the plane, and at right angles. (13)

Art. 302. (14) Ex. 12. (15) Resolve along the plane, friction acting down the plane. (16) On the smooth plane P : W : : 3 : 5 $\therefore \sin i = \frac{3}{5}$, and $\cos i = \frac{4}{5}$; resolve along the plane and at right angles. (17) Equation of Art. 315. (18) Art. 255. (19) Exam. 2. (20) Art. 315. (21) Art. 255. (22) Art. 315. (23) Equation of Art. 321. (24) Art. 287. (25) Find acceleration by Art. 313, and then use equations of Art. 14. (26) $\mu = \tan 30^\circ = \frac{1}{\sqrt{3}}$; find accelerations by Art. 313, and use equations of Art. 14. The accelerations are different on the different planes. (27) $s = \frac{u^2}{2g \sin i}$. (Art. 43). (28) From Art. 47 the general equation for the space described is $s = \frac{u^2}{2g}$, which is obtained by putting $v = 0$ in equation 3. For the smooth inclined plane g becomes $g \sin i$, and for the rough plane $g(\sin i + \mu \cos i)$.

ANSWERS.

Exercises to Chap. II. Page 13.

(1.) (*a*) 200 ft. per sec., (*b*) 250 ft., (*c*) 40 ft. per sec. (2.) 4 secs. (3.) 15 secs. (4.) (*a*) 400 ft., (*b*) 625 ft. (5.) (*a*) -200, (*b*) 375. (6.) 40 ft. (7.) 4 secs. (8.) 24 ft.; 432 ft. (9.) 30 ft. (10.) 15 ft.; 45 ft. (11.) 87 ft. (12.) 450 ft. per sec. (13.) 40 ft. per sec. (14.) (*a*) 5 secs.; (*b*) 372·5 metres. (15.) 100 metres nearly. (16.) 11,760 metres per sec. (17.) $62\frac{1}{2}$ ft.; $87\frac{1}{2}$ ft. (18.) 320 ft. (19.) 1 sec.; 3 secs. (20.) 100 ft. (21.) 48 ft. per sec. (22.) 1·65 secs. (23.) 5 secs. (24.) 6 secs. (25.) 25 ft. (27.) 6 secs. (28.) 352. (29.) $\frac{22}{75}$. (30.) 44. (31.) 38,400.

Exercises to Chap. III. Page 25.

(1.) No. (2.) 4·6 ft. per sec. (3.) 10 miles per hr. (4.) 400 ft. from A. (5.) 50 ft.; $50\sqrt{3}$ ft. (6.) $250\sqrt{3}$ ft. (7.) 100 yds. down stream; 10 mins. (8.) 8 miles and 12 miles per hr. (9.) 1 mile per hr. (10.) 10 ft. per sec.; 30°. (11.) $25\sqrt{3}$ ft. (12.) 7·8 miles. (13.) $1\frac{1}{2}$ miles. (14.) 24·16 ft. per sec. (15.) (*a*) 193·2, (*b*) 100. (16.) 5 : 264. (17.) 22 : 75. (18.) 2 ft. per sec. (19.) 50 ft. per sec. (20.) 13·3 ft. per sec. (21.) 23·8 ft. per sec. (22.) 57·1 miles. (23.) 2,500 ft. (24.) 65·68 metres; 131·3 metres. (25.) $25\sqrt{2}$ ft.; $50\sqrt{2}$ ft.; 100 ft. (26.) 24·16 ft.; 72·48 ft. (27.) 19·08 ft. (28.) 17·7. (29.) 40 ft.; 2,000 ft. (30.) 27·32 ft. per sec.; 136·6 ft.

Exercises to Chap. IV. Page 37.

(1.) 3 secs.; 2 secs. (2.) (*a*) 128 ft., (*b*) 80 ft., (*c*) 0 ft., (*d*) -640 ft., (*e*) -1152 ft. (3.) 80 ft. from top. (4.) Half-way. (5.) (*a*) 225 ft., (*b*) 200 ft., (*c*) -400 ft. (6.) $7\frac{1}{2}$

secs. (7.) 1·17 secs.; 4 secs. (8.) $64\sqrt{2}$ ft.; $-64\sqrt{2}$ ft. (9.) 12·8 ft. per sec. (10.) 200 ft. (11.) 4,096 ft. (12.) 4 secs. (13.) (*a*) 34 ft., (*b*) $15\frac{5}{8}$ secs. (14.) 15 ft. per sec. (15.) $(50+48\sqrt{3})$ ft.; $(150+72\sqrt{3})$ ft. (16.) 14,400 ft. (17.) 1,896 ft. (18.) 4 secs.; 256 ft. (19.) $8\frac{1}{2}$ secs. from starting of 1st. (20.) $3\frac{2}{9}$ secs. after starting of 1st. (21.) $\sqrt{2\sqrt{2}}$ secs. (22.) $\sqrt[4]{3}$ secs. (23.) $10\sqrt{3}$ ft. (24.) 188·6 ft.; 640·3 ft. (25.) 83·1 ft. (26.) 128 ft. (27.) 800 ft. (28.) (*a*) 1,300 ft., (*b*) 900 ft. (29.) 400 ft. (30.) 100 ft.; 250 ft.; 269·2 ft. (32.) (*a*) Vertically downwards, (*b*) 44 ft., (*c*) 16 ft. (33.) Vertically. (34.) (*a*) 100 ft., (*b*) 5 secs., (*c*) 250 ft. (35.) 3·7 secs. (36.) 2·73 secs. (37.) 13·6 secs. (38.) (*a*) Half the time taken to fall through the whole height, (*b*) at one-fourth the whole height from the top. (39.) 2 secs. nearly. (40.) (*a*) 36 ft., (*b*) 75 ft., (*c*) 3 secs. (41.) 8 ft.; -120 feet. (42.) 68·17 ft.; 306·7 ft. (43.) 1·38 secs. and 3·66 secs. (44.) $12\frac{1}{2}$ secs. (45.) 192 ft. per sec. (47.) (*a*) At the point where it was dropped, (*b*) 9·17 secs. (48.) 22. (49.) 192,000.

Exercises to Chap. V. Page 50.

(1.) (*a*) 12 ft., (*b*) $\sqrt{3}$ secs., (*c*) $66\sqrt{3}$ ft. (2.) 56 ft. (3.) $180\sqrt{5}$ ft. (4.) 2 secs. each; distances, 160 ft., 200 ft., 240 ft. (5.) $128\sqrt{2}$ ft. (6.) 400 ft. (7.) 7,500 ft. (8.) 16 ft.; 2 secs. (9.) 128 ft. (10.) 1,600 ft. (11.) 3,200 ft. (12.) Each in 2 secs. (13.) $\frac{u}{g}(\sin a+\cos a)$. (14.) 144 ft.; 64 ft. (15.) $144\sqrt{3}$ ft.; 288 ft.; 144 ft.; 0 ft. (16.) 128 ft. (17.) 256 ft.

Exercises to Chap. VI. Page 64.

(1.) 8 ft. per sec. (2.) 2 ft. per sec.; 25 ft. (3.) $\sqrt{40}$ ft. (4.) 6 ft. (5.) 16 ft. (6.) 48 ft.; 192 ft. (7.) $1\frac{1}{4}$ lb. (8.) $\frac{1}{2}$. (9.) 32; 8; 128. (10.) 10 ft.; 100 ft. (11.) $2\frac{1}{2}$ lbs. (12.) 6 ft.; 9·375 lbs.; $9·375g$ lbs. (13.) $4\frac{1}{4}$ secs.

(14.) $66\frac{2}{3}$ ft. (15.) $19\frac{1}{21}$ ozs. (16.) $3\frac{1}{3}$ lbs. (17.) $3\frac{5}{9}$ ft (18.) $4\frac{13}{16}$ lbs. (19.) 10 lbs. (20.) 32 ft. (21.) 67·136 ft. per sec. (22.) 134·27 ft. (23.) 40 miles per hr.; $\frac{4}{9}$ miles. (24.) 11 : 1080. (25.) 9·23 ft. (26.) 8·6 secs. (27.) 125 ft. (28.) $1\frac{1}{3}$ lbs. (29.) $2\frac{1}{3}$ secs. (30.) 32 ft. (31.) 7 secs. (34) 48. (35.) Mass of 100 lbs. (36.) $\frac{7}{120}$ lb. weight.

Exercises to Chap. VII. Page 74.

(1.) 15 lbs. (2.) 20 lbs. (3.) $5\frac{5}{9}$ lbs. (4.) 44·6 ft. per sec. (5.) 40 ft. (6.) $\frac{1}{2}$ ton; $\frac{21}{32}$ ton. (7.) $54\frac{6}{11}$ grammes; $54\frac{6}{11}$ g, or 53,509 dynes. (8.) $2\frac{7}{20}$ tons. (9.) $5\frac{164}{167}$ secs. (10.) (*a*) 100 lbs., (*b*) 150 lbs., (*c*) 50 lbs., (*d*) 25 lbs., (*e*) 0. (11.) (*a*) 140 lbs., (*b*) 140 lbs., (*c*) 154 lbs., (*d*) 126 lbs., (*e*) 0, (*f*) 175 lbs., (*g*) 280 lbs. (12.) (*a*) $112\frac{1}{2}$ lbs., (*b*) $87\frac{1}{2}$ lbs. (13.) 0. (14.) (*a*) 5 tons, (*b*) $6\frac{1}{4}$ tons. (15.) $4\frac{2}{7}$ ozs. (16.) $8\frac{4}{7}$ ozs. (18.) $\frac{1}{17}$ lb. (19.) 27 ozs. (20.) $T = \frac{PQ(\sin i + 1)}{P+Q}$.

Exercises to Chap. VIII. Page 84.

(1.) 4 ft. (2.) 50 ft. per sec. (3.) 20 ft. per sec. (4.) $\frac{20}{2}$, $\frac{20}{3}$, $\frac{20}{4}$, $\frac{20}{5}$, $\frac{20}{6}$, &c. (5.) $4\frac{1}{2}$ ft.; $6\frac{1}{2}$ ft. (6.) -20 ft.; -10 ft. (7.) (*a*) 0, (*b*) -4 ft.; 16 ft. (8.) $\frac{1}{4}$. (9.) 15 ft.; 25 ft. (10.) $\frac{3}{8}$. (11.) $2\frac{1}{2}$ ft.; $7\frac{1}{2}$ ft. (12.) 20 ft. per sec. (13.) 90°. (14.) 0°. (15.) 3 ft. (16.) $1\frac{1}{2}$ ft.; $\frac{3}{8}$ ft. (17.) 25 ft. per sec. (18.) 30°. (19.) 100 ft. nearly. (20.) 0.

Exercises to Chap. IX. Page 96.

(1.) 6,720 foot-pounds; 6,720 g foot-poundals. (2.) 56 H.P. (3.) $2\frac{1}{2}$ hrs. (4.) 1,344,000 ft.-pounds; 1,344,000 g ft.-poundals. (5.) 396,000 galls. (6.) 2,640 cubic ft. (7.) 450,000 ft.-pounds. (8.) 3,000,000 ft.-pounds. (9.) 31·259 g ft.-pounds. (10.) 2,897 cub. ft. (11.) $\frac{5}{99}$ H.P. (12.) $6\frac{26}{33}$ H.P. (13.) $\frac{25}{132}$. (15.) $\frac{31}{32}$ lb. (16.) $1\frac{1}{3}$ lb. (17) $1\frac{5}{18}$ lb. (18.) $2\frac{1}{2}$ secs.; 50 ft. (19.) 16 lbs. (20.) 112.

ft. (23.) $37\frac{1}{2}$ secs.; 1,800 ft. (24.) $18\frac{3}{4}$ secs.; $562\frac{1}{2}$ ft. (25.) 180 lbs. (26.) 48 H.P. (27.) 137·6 H.P. (28.) 3764·4 ft. (29.) 3,962 ft. (30.) 470·4 H.P. (31.) (*a*) $164\frac{1}{16}$ secs., (*b*) $67\frac{1}{2}$ ft. (32.) (*a*) 210 secs., (*b*) 120 ft. (33.) 716,800 foot-poundals. (34.) 20,000 foot-poundals. (35.) (*a*) 250,000 centimetre grammes; (*b*) 245,000,000 ergs. (36.) 5,000,000 ergs. (37.) $9\frac{3}{8}$ miles per hour.

Exercises to Chap. X. Page 104.

(1.) $2\sqrt{g}$ ft. per sec. (2). 3·4 lbs. (3.) $\frac{\pi^2}{9}$ lbs. (4.) 24 ft. per sec. (5.) ·484 lbs. (6.) $\frac{5}{8}\pi^2$ lbs. (11.) 16 feet per sec. (12.) 100 lbs. (13.) 338·8 lbs.

Exercises to Chap. XI. Page 115.

(1.) 39·1393 inches. (2.) 2·446 inches. (3.) 144 : 145 nearly. (4.) 55·2 secs. (5.) 64·8 secs. (6.) 1 : ·9997 (7.) 11,745 feet. (8.) 39·151 inches. (9.) 135 secs.

Exercises to Chap. XII. Page 125.

(1.) 9 lbs., 12 lbs. (2.) No. (3.) In the same line. (4.) $1 : \sqrt{2}$. (5.) $1 : \sqrt{3}$. (6.) 20 lbs. (7.) 90°. (8.) 20 lbs. (9.) $15\sqrt{3}$ lbs. (10.) $4\sqrt{10}$ lbs. (11.) 11 lbs. (12.) $\sqrt{284}$ lbs. (13.) 4·6 lbs. (14.) $\sqrt{2} : 1$. (15.) 20 lbs. (16.) $20\sqrt{2}$ lbs. (17.) 60°. (18.) $10\sqrt{3}$ lbs. (21.) 12 lbs.; 16 lbs. (22.) $3\sqrt{5}$ lbs. (23.) $\sqrt{124}$ ozs. (24.) 20 ozs. (25.) 9·2 lbs. (28.) $50\sqrt{3}$, 50. (29.) $\sqrt{3}$ lbs., 90°. (30.) 7 lbs. (31.) 25 lbs. (32.) 10·39 lbs. (33.) A, $10\sqrt{3}$; B and C, $10\sqrt{2-\sqrt{3}}$. (34.) A, $10\sqrt{2}$; B and C, $10\sqrt{2-\sqrt{2}}$. (35.) $80\sqrt{3}$ lbs. (36.) $2\sqrt{2}$ lbs. (37.) 24 lbs., 32 lbs. (38.) Twice the side. (39.) $\sqrt{77}$ lbs. (40.) 10 lbs. (41.) 29·14 lbs. (43.) $8\sqrt{3}$; $16\sqrt{3}$ lbs.

EXERCISES TO CHAP. XIII. Page 134.

(1.) 16; $2\frac{1}{2}$ ft. from one end. (2.) $17\frac{1}{2}$ inches from one end; 12 lbs. (3.) 4 lbs.; 3 ft. from greater force. (4.) 3 lbs.; 14 inches from greater. (5.) 12 lbs.; $2\frac{1}{2}$ ft. from DC; $4\frac{1}{6}$ ft. from AD. (6.) $74\frac{2}{3}$ lbs.; $149\frac{1}{3}$ lbs. (9.) 5 lbs., 3 lbs. (10.) 7 lbs., 4 lbs.

EXERCISES TO CHAP. XIV. Page 146.

(4.) 12 inches from the unoccupied angle. (5.) 14 lbs.; $7\frac{3}{7}$ ft. (6.) 19 lbs.; $5\frac{13}{19}$ ft. (7.) 12 lbs.; 7 in. (9.) $\frac{W\sqrt{3}}{4}$. (11.) 3:1. (12.) 5 in. from one end. (13.) $25\sqrt{3}$; 50; 75 lbs. (14.) $8\sqrt{3}$; 16; 24 lbs. (15.) 10 ozs. (16.) $25\sqrt{2}$; 25 lbs. (18.) 11·25; 27·4.

EXERCISES TO CHAP. XV. Page 165.

(6.) $4\sqrt{3}$ ft. (7.) 80 ft. (8.) 20 lbs. (9.) 30 lbs. (10.) 4 in. and 6 in. from adjacent sides. (11.) 6 ft. from smaller end. (12.) $7\frac{3}{7}$ ft. from end. (13.) 9 ft. (14.) $\frac{5}{18}$ h. (15.) 4 in. (16.) $\frac{4a}{7}$. (17.) $\frac{2}{9}$th bisecting line from base. (18.) $\frac{1}{12}$th diameter from centre. (20.) $5\sqrt{2}$ in. (21.) 2 ft.; 3 ft. (22.) $\frac{15}{4+\sqrt{3}}$ from adjacent side of square. (24.) 5 in. (25.) $1\frac{1}{9}$ ft. from centre. (26.) $\frac{5}{6}$ ft. from centre. (27.) $\frac{1}{40}$ diagonal $=\frac{\sqrt{2}}{10}$ inches from centre. (28.) On perpendicular, $\frac{1}{3}$rd its length from base. (29.) $\frac{17}{43}$ in. from common base. (30.) $\frac{a(2+\sqrt{2})}{9}$. (31.) $\frac{2a}{7}$. (32.) $\frac{1}{3}$ a. (33.) $\frac{11}{14}$ in. from base. (34.) $14\frac{7}{12}$ lbs. (35.) $\frac{8}{9}$ ft.; $2\frac{2}{9}$ ft. from adjacent sides.

Exercises to Chap. XVI. Page 190.

(3.) 3 : 4. (5.) $49\frac{7}{9}$ lbs.; $62\frac{2}{9}$ lbs. (6.) 4 ft. from one end. (8.) 20 lbs.; $20\frac{1}{5}$ lbs. (9.) (*a*) $12\frac{1}{2}$ lbs., (*b*) $13\frac{3}{8}$ lbs. (10.) 4. (12.) $W=P(2^n-1)$. (13.) 84; 140 lbs. (14.) Loses $10\frac{2}{11}$ lbs. (16.) 160 lbs. (17.) 8 ft. (19.) 14·4 lbs. (20.) $1\frac{4}{5}$ ft. from centre. (21.) $16\frac{2}{3}$ ozs. (22.) 22 in. from fulcrum. (23.) $1\frac{2}{3}$ ft.; 18 lbs. (24.) 40 lbs. (25.) (*a*) 1,000 lbs., (*b*) 3,240 lbs. (26.) 448 lbs. (28.) 56 lbs. (29.) 35. (30.) Loss 1 lb. per cwt. (31.) 960 lbs. (32.) $627\frac{1}{5}$ lbs; $653\frac{1}{3}$ lbs. (33.) 3 : 5. (34.) 60 lbs. (35.) $1\frac{2}{3}$ lbs. (36.) 1·5 lbs. nearly. (38.) 3 lbs. (39.) 56 lbs. (40.) 224 lbs. (41.) 154 lbs. (42.) $152\frac{1}{4}$ lbs. (43.) 400 lbs. (44.) $14\frac{1}{2}$ lbs. (45.) 5 lbs. (46.) 27. (47.) 501 lbs. (48.) System with one string and five movable pulleys. (49.) $199\frac{1}{2}$ ozs. (50.) 1 lb. (51.) 3,024 lbs. (52.) 15 lbs., 25 lbs. (53.) 40 lbs. (54.) $1{,}234\frac{2}{7}$ lbs. (55.) ·396 lb. (56.) 1,875 lbs. (57.) 10,000 lbs. (58.) $16\frac{2}{3}$ ft.

Exercises to Chap. XVII. Page 207.

(1.) $\frac{1}{\sqrt{3}}$; 1. (2.) Angle whose tangent is $\frac{5}{8}$. (3.) 188·2 lbs.; 217·4 lbs. (4.) $\frac{1}{5}$. (5.) Angle whose tangent is $\frac{1}{6}$. (6.) $\frac{1}{2\sqrt{3}}$. (7.) $7\frac{2}{9}$ ft. (8.) Over 57·7 lbs. (9.) 30°; 50 lbs. (10.) $W\frac{\sin i+\mu\cos i}{\cos\theta+\mu\sin\theta}$ and $W\frac{\sin i-\mu\cos i}{\cos\theta-\mu\sin\theta}$. (11.) $\frac{5}{7}$ths length of ladder. (12.) 25 lbs.; $25\sqrt{3}$ lbs. (13.) 50 lbs.; vertically upwards. (14.) 20 lbs. (15.) Over 40 lbs. (16.) Over 34 lbs. (17.) 202·5 lbs. (18.) 200 lbs. (19.) 4 ft. (20.) 22·8 lbs. (21.) 20 lbs. (22.) 55·7 lbs. (23.) 1,104 lbs. (24.) 2,880 lbs. (25.) 106·7 ft.; 71·1 ft. (26.) 36·9 ft.; 156·2 ft. (27.) 312·5 ft. (28.) (1) $\frac{u^2}{2g\sin i}$. (2) $\frac{u^2}{2g(\sin i+\mu\cos i}$.

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